

NLDV COMPUTER SESSION XI
Using MAPLE to analyse Andronov-Hopf
bifurcation in planar ODEs

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This session is devoted to symbolic computation of the *first Lyapunov coefficient* l_1 for Andronov-Hopf bifurcation in planar systems

$$\dot{x} = F(x, \alpha), \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R}. \quad (0.1)$$

The algorithm for computing l_1 can be summarized as follows.

Step 0: Find a critical parameter value α_0 for which the system has an equilibrium x_0 with eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$. This can be achieved, for example, by solving the nonlinear system

$$\begin{cases} F(x, \alpha) = 0, \\ \text{Tr } F_x(x, \alpha) = 0 \end{cases}$$

for (x_0, α_0) and verifying that $\det F_x(x_0, \alpha_0) = \omega_0^2 > 0$.

Step 1: Compute complex vectors $q, p \in \mathbb{C}^2$ such that

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \quad \langle p, q \rangle = 1,$$

where $A = F_x(x_0, \alpha_0)$ and $\langle p, q \rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2$.

Step 2: Compose the expression

$$H(z, \bar{z}) = \langle p, F(x_0 + zq + \bar{z}\bar{q}, \alpha_0) \rangle$$

and find the coefficients g_{20} , g_{11} , and g_{21} in the formal Taylor series

$$H(z, \bar{z}) = i\omega_0 z + \sum_{j+k \geq 2} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k.$$

Step 3: Compute the first Lyapunov coefficient:

$$l_1 = \frac{1}{2\omega_0^2} \text{Re} (ig_{20}g_{11} + \omega_0 g_{21}).$$

If $l_1 < 0$ ($l_1 > 0$), then the equilibrium x_0 is stable (unstable) at α_0 and a unique stable (unstable) limit cycle bifurcates from x_0 in (0.1) as α passes α_0 , provided that

$$\left. \frac{d}{d\alpha} \text{Tr } F_x(x_0(\alpha), \alpha) \right|_{\alpha=\alpha_0} \neq 0.$$

In this session we use the symbolic manipulation software MAPLE that can be started, for example, by the menu command

Start->Standard Applications->Mathematics->Maple 15->
Classic Worksheet Maple 15

1 Andronov-Hopf bifurcation in the Brusselator

Consider the following chemical model (*Brusselator*):

$$\begin{cases} \dot{x}_1 &= A - (B + 1)x_1 + x_1^2 x_2 \equiv F_1(x_1, x_2, A, B), \\ \dot{x}_2 &= Bx_1 - x_1^2 x_2 \equiv F_2(x_1, x_2, A, B). \end{cases} \quad (1.1)$$

We fix $A > 0$ and take B as a bifurcation parameter. The following sequence of MAPLE commands proves that at $B = 1 + A^2$ the system exhibits a supercritical Andronov-Hopf bifurcation. Note that output details can depend on the MAPLE version and computer used.

```
> with(linalg):
```

This command allows us to use the MAPLE linear algebra package.

```
> F[1]:=A-(B+1)*X[1]+X[1]^2*X[2];
      F1 := A - (B + 1) X1 + X1^2 X2
```

```
> F[2]:=B*X[1]-X[1]^2*X[2];
      F2 := B X1 - X1^2 X2
```

```
> J:=jacobian([F[1],F[2]],[X[1],X[2]]);
      J := \begin{bmatrix} -B - 1 + 2 X_1 X_2 & X_1^2 \\ B - 2 X_1 X_2 & -X_1^2 \end{bmatrix}
```

```
> K:=transpose(J);
      K := \begin{bmatrix} -B - 1 + 2 X_1 X_2 & B - 2 X_1 X_2 \\ X_1^2 & -X_1^2 \end{bmatrix}
```

By these commands we input the right-hand sides of (1.1) into MAPLE and compute its Jacobian matrix J and its transpose K .

```
> sol:=solve({F[1]=0,F[2]=0},{X[1],X[2]});
      sol := \left\{ X_1 = A, X_2 = \frac{B}{A} \right\}
```

```
> assign(sol);
```

By this we have found the unique equilibrium of (1.1):

$$X = \left(A, \frac{B}{A} \right)^T$$

as function of A and B .

```
> T:=eval(trace(J));
```

$$T := B - 1 - A^2$$

```
> diff(T,B);
```

$$1$$

```
> sol:=solve({T=0},{B});
```

$$sol := \{ B = 1 + A^2 \}$$

```
> assign(sol);
```

These commands show that at $B = 1 + A^2$ the trace of J evaluated at the equilibrium X vanishes: $\text{tr } J = 0$. This means that J has two eigenvalues with $\lambda_1 + \lambda_2 = 0$. Moreover, the derivative of $\text{tr } J$ with respect to the parameter B is always nonzero (equals to 1).

```
> assume(A>0);
```

```
> omega:=sqrt(det(J));
```

$$\omega := A$$

Thus, at $B = 1 + A^2$ the Brusselator has the equilibrium

$$X = \left(A, \frac{1 + A^2}{A} \right)^T$$

with purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$, $\omega = \sqrt{\det(J)} = A > 0$. Therefore, the Andronov-Hopf bifurcation condition is satisfied. To apply the theoretical results on the bifurcation, we have to verify that

(i) the eigenvalues cross the imaginary axes with a nonzero velocity with respect to parameter B ;

(ii) the first Lyapunov coefficient l_1 does not vanish.

The first genericity condition has been already verified, since $\text{Tr } J = 2\text{Re } \lambda_{1,2}$ and the derivative of the trace with respect to B is equal to 1. Moreover, the equilibrium

is stable for $B < 1 + A^2$ and unstable for $B > 1 + A^2$. The rest of the section is devoted to the computation of l_1 .

```
> ev:=eigenvectors(map(eval,J));
ev :=  $\left[ A^\sim I, 1, \left\{ \left[ 1, -\frac{-A^\sim I + A^{\sim 2}}{A^{\sim 2}} \right] \right\}, \left[ -I A^\sim, 1, \left\{ \left[ 1, -\frac{A^\sim I + A^{\sim 2}}{A^{\sim 2}} \right] \right\} \right] \right]$ 

> q:=ev[1][3][1];
q :=  $\left[ 1, -\frac{-A^\sim I + A^{\sim 2}}{1 + A^{\sim 2}} \right]$ 

> q:=evalm(1/q[2]*q): q:=map(simplify,map(evalc,q));
q :=  $\left[ -\frac{A^\sim(A^\sim + I)}{1 + A^{\sim 2}}, 1 \right]$ 

> et:=eigenvectors(map(eval,K));
et :=  $\left[ A^\sim I, 1, \left\{ \left[ \frac{A^\sim I + A^{\sim 2}}{A^{\sim 2}}, 1 \right] \right\}, \left[ -I A^\sim, 1, \left\{ \left[ \frac{-A^\sim I + A^{\sim 2}}{A^{\sim 2}}, 1 \right] \right\} \right] \right]$ 

> p:=et[2][3][1];
p :=  $\left[ \frac{-A^\sim I + A^{\sim 2}}{A^{\sim 2}}, 1 \right]$ 
```

These commands compute

$$q = \left(-\frac{iA + A^2}{1 + A^2}, 1 \right)^T, \quad p = \left(\frac{-iA + A^2}{A^2}, 1 \right)^T,$$

the critical eigenvectors of the Jacobian matrix J and its transpose, so that

$$Jq = i\omega q, \quad J^T p = -i\omega p.$$

```
> s1:=simplify(evalc(conjugate(p[1])*q[1]+conjugate(p[2])*q[2]));
s1 :=  $-\frac{2(-1 + A^\sim I)}{1 + A^{\sim 2}}$ 

> p:=evalm(1/conjugate(s1)*p): p:=map(simplify,map(evalc,p));
p :=  $\left[ \frac{\frac{-1}{2} I (1 + A^{\sim 2})}{A^\sim}, \frac{1}{2} - \frac{1}{2} I A^\sim \right]$ 

> simplify(evalc(conjugate(p[1])*q[1]+conjugate(p[2])*q[2]));
1
```

By the commands above, we achieve the normalization $\langle p, q \rangle = 1$, finally taking

$$p = \left(-\frac{i(1+A^2)}{2A}, \frac{1-iA}{2} \right)^T.$$

```
> F[1]:=A-(B+1)*x[1]+x[1]^2*x[2];
      F1 := A~ - (2 + A~^2) x1 + x1^2 x2

> F[2]:=B*x[1]-x[1]^2*x[2];
      F2 := (1 + A~^2) x1 - x1^2 x2

> x[1]:=evalc(X[1]+z*q[1]+z1*conjugate(q[1]));
      x1 := A~ - \frac{z A~^2}{1 + A~^2} - \frac{z I A~^2}{1 + A~^2} + \left( -\frac{z A~}{1 + A~^2} + \frac{z I A~}{1 + A~^2} \right) I

> x[2]:=evalc(X[2]+z*q[2]+z1*conjugate(q[2]));
      x2 := \frac{1}{A~} + A~ + z + z I

> H:=simplify(evalc(conjugate(p[1])*F[1]+conjugate(p[2])*F[2]));
      H := -\frac{1}{2}(-z I^2 A~^2 z + 2 z z I A~ + 3 z^2 A~^2 z I + I z I^2 - 2 z I^2 A~^3
      - 2 z^2 A~^3 + I z^2 - z^2 A~ - z I^2 A~ + z^3 A~^2 - 3 z I^3 A~^2 + 3 A~^4 z^2 z I
      + 3 A~^4 z I^2 z - 2 z A~^5 z I + 2 I z^2 A~^2 - 2 I z z I - 2 I z A~
      + I z I^2 A~^4 + I z^3 A~ + I z I^3 A~ - 2 I z A~^5 - 3 I z I^3 A~^3 + I z^3 A~^3
      - 4 I z A~^3 + I z^2 A~^4 + 2 I z I^2 A~^2 - I z^2 A~ z I - 5 I z I^2 A~^3 z
      - I z I^2 A~ z + 2 I z A~^4 z I - I z^2 A~^3 z I + A~^4 z^3 + A~^4 z I^3
      - z I^2 A~^5 - z^2 A~^5) / (1 + A~^2)^2
```

By means of these commands, we set $x = X + zq + \bar{z}\bar{q}$ and compose the function

$$H(z, \bar{z}) = \langle p, F(X + zq + \bar{z}\bar{q}, A, 1 + A^2) \rangle.$$

(In our MAPLE input, z1 stands for $\bar{z}$.)

```
> g[2,0]:=simplify(2*evalc(coeftayl(H,[z,z1]=[0,0],[2,0])));
      g2,0 := A~ - I
```

```
> g[1,1]:=simplify(evalc(coeftayl(H,[z,z1]=[0,0],[1,1])));
```

$$g_{1,1} := \frac{(A^2 - 1)(A - I)}{1 + A^2}$$

```
> g[2,1]:=simplify(2*evalc(coeftayl(H,[z,z1]=[0,0],[2,1])));
```

$$g_{2,1} := -\frac{A(3A - I)}{1 + A^2}$$

The above commands compute the coefficients g_{jk} of the Taylor expansion of $H(z, \bar{z})$ at $(z, \bar{z}) = (0, 0)$,

$$H(z, \bar{z}) = i\omega z + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4),$$

giving

$$g_{20} = A - i, \quad g_{11} = \frac{(A - i)(A^2 - 1)}{1 + A^2}, \quad g_{21} = \frac{A(i - 3A)}{1 + A^2}.$$

```
> l[1]:=simplify(1/(2*omega^2)*Re(I*g[2,0]*g[1,1]+omega*g[2,1]));
```

$$l_1 := -\frac{2 + A^2}{2A(1 + A^2)}$$

This final command computes the first Lyapunov coefficient

$$l_1 = \frac{1}{2\omega^2} \text{Re}(ig_{20}g_{11} + \omega g_{2,1}) = -\frac{2 + A^2}{2A(1 + A^2)} < 0,$$

and allows us to verify that it is negative for all $A > 0$.

2 Andronov-Hopf bifurcation in a predator-prey model

Compute the first Lyapunov coefficient for the Andronov-Hopf bifurcation in the system:

$$\begin{aligned} \dot{x}_1 &= rx_1(1 - x_1) - \frac{cx_1x_2}{\alpha + x_1}, \\ \dot{x}_2 &= -dx_2 + \frac{cx_1x_2}{\alpha + x_1}. \end{aligned}$$

The system describes dynamics of a simple predator-prey ecosystem. Here x_1 and x_2 are (scaled) population numbers, and r, c, d , and α are positive parameters characterizing the behavior of isolated populations and their interaction. Consider α as a control parameter and assume $c > d$.

2.1 Steps of the analysis

(i) Consider an orbitally equivalent polynomial system

$$\begin{cases} \dot{x}_1 &= r x_1 (\alpha + x_1) (1 - x_1) - c x_1 x_2 \equiv F_1(x_1, x_2, \alpha) \\ \dot{x}_2 &= -\alpha d x_2 + (c - d) x_1 x_2 \equiv F_2(x_1, x_2, \alpha). \end{cases} \quad (2.1)$$

(ii) Show that at the parameter value

$$\alpha_0 = \frac{c - d}{c + d}$$

the system (2.1) has an equilibrium x_0 with eigenvalues $\lambda_{1,2} = \pm i\omega, \omega > 0$, where

$$\omega^2 = \frac{rc^2d(c - d)}{(c + d)^3}.$$

(iii) Write the Jacobian matrix of (2.1) at x_0 for α_0 in the form ¹

$$A = \begin{pmatrix} 0 & -\frac{cd}{c + d} \\ \frac{\omega^2(c + d)}{cd} & 0 \end{pmatrix},$$

and find complex eigenvectors q, p of A and A^T corresponding to $i\omega$ and $-i\omega$, respectively, and satisfying $\langle p, q \rangle = 1$.

(iv) Compute the coefficients g_{20}, g_{11} , and g_{21} of the Taylor expansion of the function $g = \langle p, F(x_0 + zq + \bar{z}\bar{q}, \alpha_0) \rangle$ with respect to $z, \bar{z}$ at $z = \bar{z} = 0$, and check that

$$l_1 = \frac{1}{2\omega^2} \operatorname{Re}(ig_{20}g_{11} + \omega g_{21}) < 0.$$

Conclude that a unique and stable limit cycle bifurcates from the equilibrium via the Andronov-Hopf bifurcation for $\alpha < \alpha_0$

2.2 MAPLE session

```
> with(linalg):
> F[1]:=r*X[1]*(alpha+X[1])*(1-X[1])-c*X[1]*X[2];
      F1:=r X1 (alpha+X1) (1-X1)-c X1 X2

> F[2]:=-alpha*d*X[2]+(c-d)*X[1]*X[2];
      F2:=-alpha d X2+(c-d) X1 X2
```

¹It is always useful to express the Jacobian matrix using ω , since this simplifies expressions for the eigenvectors.

> J:=jacobian([F[1],F[2]], [X[1],X[2]]);

$$J := \begin{bmatrix} r(\alpha + X_1)(1 - X_1) + rX_1(1 - X_1) - rX_1(\alpha + X_1) - cX_2, & -cX_1 \\ (c - d)X_2, & -\alpha d + (c - d)X_1 \end{bmatrix}$$

> K:=transpose(J);

$$K := \begin{bmatrix} r(\alpha + X_1)(1 - X_1) + rX_1(1 - X_1) - rX_1(\alpha + X_1) - cX_2, \\ (c - d)X_2 \\ [-cX_1, -\alpha d + (c - d)X_1] \end{bmatrix}$$

> sol:=solve({F[1]=0,F[2]=0,trace(J)=0},{X[1],X[2],alpha});

$$sol := \{X_2 = 0, X_1 = 0, \alpha = 0\}, \left\{X_1 = \frac{d}{c+d}, \alpha = \frac{c-d}{c+d}, X_2 = \frac{rc}{(c+d)^2}\right\}, \\ \left\{X_2 = 0, X_1 = 1, \alpha = \frac{-r+c-d}{r+d}\right\}, \left\{X_2 = 0, X_1 = \frac{r+c}{r}, \alpha = -\frac{r+c}{r}\right\}$$

> sol:=sol[2];

$$sol := \left\{X_1 = \frac{d}{c+d}, \alpha = \frac{c-d}{c+d}, X_2 = \frac{rc}{(c+d)^2}\right\}$$

> assign(sol);

> assume(r>0,c>0,d>0,c>d);

> J:=map(simplify,map(eval,J));

$$J := \begin{bmatrix} 0 & -\frac{c^{\sim} d^{\sim}}{c^{\sim} + d^{\sim}} \\ \frac{(c^{\sim} - d^{\sim}) r^{\sim} c^{\sim}}{(c^{\sim} + d^{\sim})^2} & 0 \end{bmatrix}$$

> DD:=simplify(det(J));

$$DD := \frac{c^{\sim 2} d^{\sim} (c^{\sim} - d^{\sim}) r^{\sim}}{(c^{\sim} + d^{\sim})^3}$$

> A:=matrix(2,2,[0,J[1,2],omega^2*J[2,1]/DD,0]);

$$A := \begin{bmatrix} 0 & -\frac{c^{\sim} d^{\sim}}{c^{\sim} + d^{\sim}} \\ \frac{\omega^2 (c^{\sim} + d^{\sim})}{c^{\sim} d^{\sim}} & 0 \end{bmatrix}$$

> det(A);

$$\omega^2$$

```

> assume(omega>0);
> ev:=eigenvects(map(eval,A));
ev :=  $\left[ I \omega^{\sim}, 1, \left\{ \left[ \frac{I c^{\sim} d^{\sim}}{\omega^{\sim} (c^{\sim} + d^{\sim})}, 1 \right] \right\} \right], \left[ -I \omega^{\sim}, 1, \left\{ \left[ -\frac{I c^{\sim} d^{\sim}}{\omega^{\sim} (c^{\sim} + d^{\sim})}, 1 \right] \right\} \right]$ 

> q:=ev[1][3][1];

$$q := \left[ \frac{I c^{\sim} d^{\sim}}{\omega^{\sim} (c^{\sim} + d^{\sim})}, 1 \right]$$


> et:=eigenvects(map(eval,transpose(A)));
et :=  $\left[ I \omega^{\sim}, 1, \left\{ \left[ -\frac{I (c^{\sim} + d^{\sim}) \omega^{\sim}}{c^{\sim} d^{\sim}}, 1 \right] \right\} \right], \left[ -I \omega^{\sim}, 1, \left\{ \left[ \frac{I (c^{\sim} + d^{\sim}) \omega^{\sim}}{c^{\sim} d^{\sim}}, 1 \right] \right\} \right]$ 

> P:=et[2][3][1];

$$P := \left[ \frac{I (c^{\sim} + d^{\sim}) \omega^{\sim}}{c^{\sim} d^{\sim}}, 1 \right]$$


> s:=simplify(evalc(conjugate(P[1])*q[1]+conjugate(P[2])*q[2]));

$$s := 2$$


> p:=evalm(1/conjugate(s)*P);

$$p := \left[ \frac{\frac{1}{2} I \omega^{\sim} (c^{\sim} + d^{\sim})}{c^{\sim} d^{\sim}}, \frac{1}{2} \right]$$


> simplify(evalc(conjugate(p[1])*q[1]+conjugate(p[2])*q[2]));

$$1$$


> F[1]:=r*x[1]*(alpha+x[1])*(1-x[1])-c*x[1]*x[2];

$$F_1 := r^{\sim} x_1 \left( \frac{c^{\sim} - d^{\sim}}{c^{\sim} + d^{\sim}} + x_1 \right) (1 - x_1) - c^{\sim} x_1 x_2$$


> F[2]:=-alpha*d*x[2]+(c-d)*x[1]*x[2];

$$F_2 := -\frac{(c^{\sim} - d^{\sim}) d^{\sim} x_2}{c^{\sim} + d^{\sim}} + (c^{\sim} - d^{\sim}) x_1 x_2$$


> x[1]:=evalc(X[1]+z*q[1]+z1*conjugate(q[1]));

$$x_1 := \frac{d^{\sim}}{c^{\sim} + d^{\sim}} + I \left( \frac{z c^{\sim} d^{\sim}}{\omega^{\sim} (c^{\sim} + d^{\sim})} - \frac{z1 c^{\sim} d^{\sim}}{\omega^{\sim} (c^{\sim} + d^{\sim})} \right)$$


> x[2]:=evalc(X[2]+z*q[2]+z1*conjugate(q[2]));

$$x_2 := \frac{r^{\sim} c^{\sim}}{(c^{\sim} + d^{\sim})^2} + z + z1$$


```

```

> H:=simplify(evalc(conjugate(p[1])*F[1]+conjugate(p[2])*F[2])):
> g[2,0]:=simplify(2*evalc(coeftayl(H,[z,z1]=[0,0],[2,0])));

$$g_{2,0} := -\left(\tilde{c}^2 \tilde{\omega} - I \tilde{c}^2 \tilde{d} + 2 \tilde{\omega} \tilde{d} \tilde{c} + I \tilde{d}^2 \tilde{r} + I \tilde{d}^3 + \tilde{\omega} \tilde{d}^2\right) \tilde{c} / \left(\tilde{\omega} (\tilde{c} + \tilde{d})^2\right)$$

> g[1,1]:=simplify(evalc(coeftayl(H,[z,z1]=[0,0],[1,1])));

$$g_{1,1} := \frac{I \tilde{r} \tilde{d}^2 \tilde{c}}{\tilde{\omega} (\tilde{c} + \tilde{d})^2}$$

> g[2,1]:=simplify(2*evalc(coeftayl(H,[z,z1]=[0,0],[2,1])));

$$g_{2,1} := -3 \frac{\tilde{r} \tilde{d}^2 \tilde{c}^2}{(\tilde{c} + \tilde{d})^2 \tilde{\omega}^2}$$

> l[1]:=factor(1/(2*omega^2)*Re(I*g[2,0]*g[1,1]+omega*g[2,1]));

$$l_1 := -\frac{\tilde{c}^2 \tilde{d}^2 \tilde{r}}{\tilde{\omega}^3 (\tilde{c} + \tilde{d})^2}$$


```

3 Additional problems

- Verify that the complex pair of eigenvalues of the positive equilibrium in (2.1) crosses the imaginary axes with nonzero velocity at the Hopf bifurcation.
- Check that each of the following systems has an equilibrium that exhibits a Hopf bifurcation at some value of parameter α , and compute the first Lyapunov coefficient. Verify also that the eigenvalues cross the imaginary axis at nonzero velocity.

(a) *Rayleigh's equation:*

$$\ddot{x} + \dot{x}^3 - 2\alpha\dot{x} + x = 0$$

(Hint: Introduce $y = \dot{x}$ and rewrite the equation as a system of two differential equations.)

(b) *Van der Pol's oscillator:*

$$\ddot{y} - (\alpha - y^2)\dot{y} + y = 0;$$

(c) *Bautin's example:*

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x + \alpha y + x^2 + xy + y^2; \end{cases}$$

(d) *Advertising diffusion model by G. Feichtinger :*

$$\begin{cases} \dot{x}_1 &= \alpha[1 - x_1x_2^2 + A(x_2 - 1)] \\ \dot{x}_2 &= x_1x_2^2 - x_2. \end{cases}$$

- Analyze Hopf bifurcation in the predator-prey model by Bazykin&Khibnik (1981):

$$\begin{aligned} \dot{x} &= \frac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -\gamma y(m-x). \end{aligned}$$

where γ, n , and m are positive parameters.

- (1) Find a relationship between parameters m and n corresponding to a Hopf bifurcation in the system.
- (2) Compute the first Lyapunov coefficient l_1 along the curve in the (m, n) -plane found at step (1) and show that it vanishes at

$$(m, n) = \left(\frac{1}{4}, \frac{1}{8} \right).$$