Applied Bifurcation Theory: Practicum 1, 15 July 2019

For each planar system below, construct its phase portrait for $\alpha = 0$ and for small $\alpha < 0$ and $\alpha > 0$ using the `MATLAB` tool `pplane9`. Identify the occurring bifurcation and try to support your conclusions by analytical arguments as outlined below.

Ex.1 Saddle-node homoclinic bifurcation

$$\left\{ \begin{array}{rcl} \dot{x} & = & x(1 - x^2 - y^2) \; - \; y(1 + \alpha + x) \\ \dot{y} & = & x(1 + \alpha + x) \; + \; y(1 - x^2 - y^2) \end{array} \right. \tag{1}$$

1. Rewrite the system in the polar coordinates (r, φ) by substituting $x = r\cos\varphi,\; y = r\sin\varphi$.
2. Prove that the unit circle $r = 1$ is invariant and study equilibria on this circle.
3. Compute the normal form coefficient a for the saddle-node (fold) bifurcation at $\alpha = 0$.

Ex.2 Andronov-Hopf bifurcation

$$\ddot{x} + \dot{x}^3 - 2\alpha\dot{x} + x = 0 \tag{2}$$

Rewrite this equation as a planar system by introducing $y = -\dot{x}$.

1. Consider the complex variable $z = x + iy$ and write the planar system for $\alpha = 0$ as one complex equation $\dot{z} = i\omega z + g(z, \bar{z})$.
2. Compute the Taylor coefficients g_{20}, g_{11}, g_{21}, and evaluate the first Lyapunov coefficient l_1.
3. Predict stability of the bifurcating cycle and the direction of its bifurcation.

Ex.3 Saddle homoclinic bifurcation

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x + 2y + x^2 \\ \dot{y} & = & (2 - \alpha)x - y - 3x^2 + \frac{3}{2}xy \end{array} \right. \tag{3}$$

1. Prove that at $\alpha = 0$ the system has an orbit homoclinic to a saddle. *Hint*: The curve $x^2(1 - x) - y^2 = 0$ is invariant, i.e. consists of orbits.
2. Predict stability of the bifurcating cycle and the direction of its bifurcation.

Ex.4 Cyclic fold bifurcation

$$\left\{ \begin{array}{rcl} \dot{x} & = & \left(\alpha - \frac{1}{4}\right)x - y + x(x^2 + y^2) - x(x^2 + y^2)^2 \\ \dot{y} & = & x + \left(\alpha - \frac{1}{4}\right)y + y(x^2 + y^2) - y(x^2 + y^2)^2 \end{array} \right. \tag{4}$$

1. Introduce polar coordinates $x = r\cos\varphi,\; y = r\sin\varphi$.
2. Analyze the number and stability of equilibria of the r-equation for varying α. Plot the equilibria (vertically) versus α (horizontally).
3. Show that $\alpha = 0$ a fold bifurcation of cycles occurs in the full planar system.

Ex.5 Saddle heteroclinic bifurcation

$$\left\{ \begin{array}{rcl} \dot{x} & = & 1 - x^2 - \alpha xy \\ \dot{y} & = & xy + \alpha(1 - x^2) \end{array} \right. \tag{5}$$

Prove that for $\alpha = 0$ there exists a heteroclinic connection between two saddles:

1. Determine the equilibria and classify them.
2. Compute the heteroclinic solution explicitly and verify the limits $t \to \pm\infty$.