Stability of Travelling Waves

Dichotomies, spectra and Fredholm properties

Arthur Vromans

arthur.j.vromans@gmail.com

May 2nd 2012

Björn Sandstede

- Professor, Division of Applied Mathematics, Brown University, Providence
- Author of *Stability of travelling waves (2002)*

Overview

Differential Equation

Waves

Overview

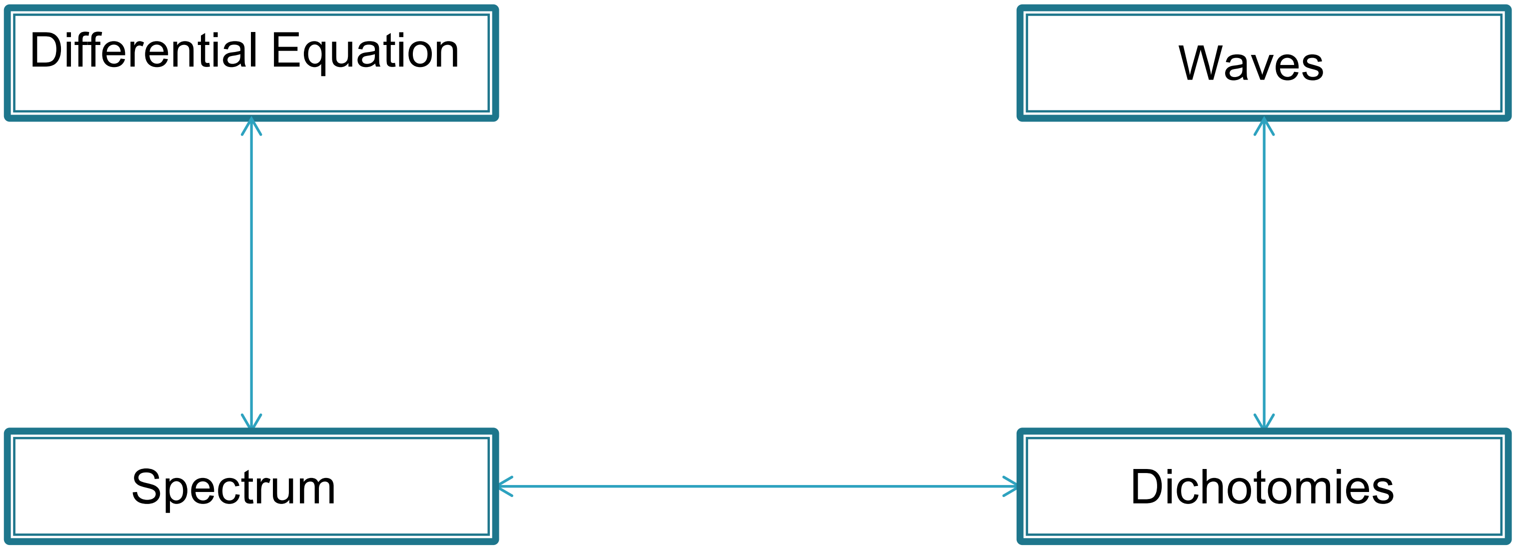
Differential Equation

Spectrum

Waves

Dichotomies

Overview



Assumptions

We consider partial differential equations of the form

$$U_t = \mathcal{A}(\partial_x)U + \mathcal{N}(U), \qquad x \in \mathbb{R}, \qquad U \in \mathcal{X}. \qquad (1)$$

Assumptions

We consider partial differential equations of the form

$$U_t = \mathcal{A}(\partial_x)U + \mathcal{N}(U), \qquad x \in \mathbb{R}, \qquad U \in \mathcal{X}. \tag{1}$$

Travelling waves are solutions of (1) of the form

$$U(x,t) = Q(x-ct) \tag{2}$$

Assumptions

We consider partial differential equations of the form

$$U_t = \mathcal{A}(\partial_x)U + \mathcal{N}(U), \qquad x \in \mathbb{R}, \qquad U \in \mathcal{X}. \qquad (1)$$

Travelling waves are solutions of (1) of the form

$$U(x,t) = Q(x-ct) \qquad (2)$$

We introduce coordinate

$$\xi = x - ct$$

Hypothesis *The matrix-valued function* $A(\xi;\lambda) \in \mathbb{C}^{n\times n}$ *is of the form*

$$A(\xi;\lambda) = \tilde{A}(\xi) + \lambda B(\xi) \tag{3}$$

where $\tilde{A}(\cdot)$ *and* $B(\cdot)$ *are in* $C^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$.

Dichotomies

We consider the eigenvalue λ dependend linearized system of (1)

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;\lambda)u \tag{4}$$

Dichotomies

We consider the eigenvalue λ dependend linearized system of (1)

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi; \lambda) u \tag{4}$$

with evolutionary operator Φ with the properties:

$$\begin{aligned} &\Phi(\xi, \zeta) = \Phi(\xi, \zeta; \lambda) \\ &\Phi(\xi, \xi) = \mathrm{id},\ \Phi(\xi, \tau)\Phi(\tau, \zeta) = \Phi(\xi, \zeta) \text{ for all } \xi, \tau, \zeta \in \mathbb{R} \\ &\quad u(\xi) = \Phi(\xi, \zeta) u_0 \text{ satisfies } (4) \text{ for every } u_0 \in \mathbb{C}^n \end{aligned} \tag{5}$$

Definition 1 **(Exponential dichotomies)** *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that* (4), *with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on I if constants $K > 0$ and $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ exist as well as a family of projections $P(\xi)$, defined and continuous for $\xi \in I$, such that the following is true for $\xi, \zeta \in I$.*

- *With $\Phi^{\mathrm{s}}(\xi, \zeta) := \Phi(\xi, \zeta) P(\zeta)$, we have*

$$|\Phi^{\mathrm{s}}(\xi, \zeta)| \leq K \mathrm{e}^{\kappa^{\mathrm{s}}(\xi - \zeta)}, \qquad \xi \geq \zeta, \qquad \xi, \zeta \in I.$$

Definition 1 **(Exponential dichotomies)** *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that (4), with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on I if constants $K > 0$ and $\kappa^s < 0 < \kappa^u$ exist as well as a family of projections $P(\xi)$, defined and continuous for $\xi \in I$, such that the following is true for $\xi, \zeta \in I$.*

- *With $\Phi^s(\xi,\zeta) := \Phi(\xi,\zeta)P(\zeta)$, we have*

$$|\Phi^s(\xi,\zeta)| \leq K e^{\kappa^s(\xi-\zeta)}, \qquad \xi \geq \zeta, \qquad \xi,\zeta \in I.$$

- *Define $\Phi^u(\xi,\zeta) := \Phi(\xi,\zeta)(\mathrm{id} - P(\zeta))$, then*

$$|\Phi^u(\xi,\zeta)| \leq K e^{\kappa^u(\xi-\zeta)}, \qquad \xi \leq \zeta, \qquad \xi,\zeta \in I.$$

Definition 1 **(Exponential dichotomies)** *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that (4), with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on I if constants $K > 0$ and $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ exist as well as a family of projections $P(\xi)$, defined and continuous for $\xi \in I$, such that the following is true for $\xi, \zeta \in I$.*

- *With $\Phi^{\mathrm{s}}(\xi,\zeta) := \Phi(\xi,\zeta)P(\zeta)$, we have*

$$|\Phi^{\mathrm{s}}(\xi,\zeta)| \le K \mathrm{e}^{\kappa^{\mathrm{s}}(\xi-\zeta)}, \qquad \xi \ge \zeta, \qquad \xi,\zeta \in I.$$

- *Define $\Phi^{\mathrm{u}}(\xi,\zeta) := \Phi(\xi,\zeta)(\mathrm{id} - P(\zeta))$, then*

$$|\Phi^{\mathrm{u}}(\xi,\zeta)| \le K \mathrm{e}^{\kappa^{\mathrm{u}}(\xi-\zeta)}, \qquad \xi \le \zeta, \qquad \xi,\zeta \in I.$$

- *The projections commute with the evolution, $\Phi(\xi,\zeta)P(\zeta) = P(\xi)\Phi(\xi,\zeta)$, so that*

$$\begin{aligned} \Phi^{\mathrm{s}}(\xi,\zeta)u_0 &\in \mathrm{R}(P(\xi)), \qquad && \xi \ge \zeta, \qquad && \xi,\zeta \in I \\ \Phi^{\mathrm{u}}(\xi,\zeta)u_0 &\in \mathrm{N}(P(\xi)), \qquad && \xi \le \zeta, \qquad && \xi,\zeta \in I. \end{aligned}$$

Definition 1 **(Exponential dichotomies)** *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that (4), with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on I if constants $K > 0$ and $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ exist as well as a family of projections $P(\xi)$, defined and continuous for $\xi \in I$, such that the following is true for $\xi, \zeta \in I$.*

- *With $\Phi^{\mathrm{s}}(\xi,\zeta) := \Phi(\xi,\zeta)P(\zeta)$, we have*

$$|\Phi^{\mathrm{s}}(\xi,\zeta)| \leq K\mathrm{e}^{\kappa^{\mathrm{s}}(\xi-\zeta)}, \qquad \xi \geq \zeta, \qquad \xi,\zeta \in I.$$

- *Define $\Phi^{\mathrm{u}}(\xi,\zeta) := \Phi(\xi,\zeta)(\mathrm{id} - P(\zeta))$, then*

$$|\Phi^{\mathrm{u}}(\xi,\zeta)| \leq K\mathrm{e}^{\kappa^{\mathrm{u}}(\xi-\zeta)}, \qquad \xi \leq \zeta, \qquad \xi,\zeta \in I.$$

- *The projections commute with the evolution, $\Phi(\xi,\zeta)P(\zeta) = P(\xi)\Phi(\xi,\zeta)$, so that*

$$\begin{aligned} \Phi^{\mathrm{s}}(\xi,\zeta)u_0 &\in \mathrm{R}(P(\xi)), \qquad && \xi \geq \zeta, \qquad && \xi,\zeta \in I \\ \Phi^{\mathrm{u}}(\xi,\zeta)u_0 &\in \mathrm{N}(P(\xi)), \qquad && \xi \leq \zeta, \qquad && \xi,\zeta \in I. \end{aligned}$$

The ξ-independent dimension of $\mathrm{N}(P(\xi))$ is referred to as the Morse index of the exponential dichotomy on I. If (4) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$, the associated Morse indices are denoted by $i_+(\lambda_*)$ and $i_-(\lambda_*)$, respectively.

Theorem 1 *Firstly, let I be $\mathbb{R}^+$ or $\mathbb{R}^-$. Suppose that $A(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ and that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi) u \tag{6}$$

has an exponential dichotomy on I with constants K, κ^{s} and κ^{u} as in Definition 1. There are then positive constants δ_* and C such that the following is true.

Theorem 1 *Firstly, let I be $\mathbb{R}^+$ or $\mathbb{R}^-$. Suppose that $A(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ and that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi) u \tag{6}$$

has an exponential dichotomy on I with constants K, κ^{s} and κ^{u} as in Definition 1. There are then positive constants δ_* and C such that the following is true. If $B(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ such that $\sup_{\xi\in I, |\xi|\geq L} |B(\xi)| < \delta/C$ for some $\delta < \delta_*$ and some $L \geq 0$, then a constant $\tilde{K} > 0$ exists such that the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = (A(\xi) + B(\xi)) u \tag{7}$$

has an exponential dichotomy on I with constants $\tilde{K}$, $\kappa^{\mathrm{s}} + \delta$ and $\kappa^{\mathrm{u}} - \delta$.

Theorem 1 *Firstly, let I be $\mathbb{R}^+$ or $\mathbb{R}^-$. Suppose that $A(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ and that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi)u \tag{6}$$

has an exponential dichotomy on I with constants K, κ^{s} and κ^{u} as in Definition 1. There are then positive constants δ_* and C such that the following is true. If $B(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ such that $\sup_{\xi\in I,|\xi|\geq L} |B(\xi)| < \delta/C$ for some $\delta < \delta_*$ and some $L \geq 0$, then a constant $\tilde{K} > 0$ exists such that the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = (A(\xi) + B(\xi))u \tag{7}$$

has an exponential dichotomy on I with constants $\tilde{K}$, $\kappa^{\mathrm{s}} + \delta$ and $\kappa^{\mathrm{u}} - \delta$. Moreover, the projections $P(\xi)$ and evolutions $\Phi^{\mathrm{s}}(\xi, \zeta)$ and $\Phi^{\mathrm{u}}(\xi, \zeta)$ associated with (7) are δ-close to those associated with (6) for all $\xi, \zeta \in I$ with $|\xi|, |\zeta| \geq L$. Secondly, if $I = \mathbb{R}$, then the above statement is true with $L = 0$.

Remark 1 *If the perturbation* $B(\xi)$ *in (7) converges to zero as* $|\xi| \to \infty$ *with* $\xi \in I$*, then the projections and evolutions of (7) converge to those of (6).*

It is also true that, if (4) *has an exponential dichotomy for* $\lambda = \lambda_*$*, then the evolutions and projections that appear in Definition 1 can be chosen to depend analytically on* λ *for* λ *close to* λ_* .

Remark 1 *If the perturbation* $B(\xi)$ *in (7) converges to zero as* $|\xi| \to \infty$ *with* $\xi \in I$*, then the projections and evolutions of (7) converge to those of (6).*

It is also true that, if (4) *has an exponential dichotomy for* $\lambda = \lambda_*$*, then the evolutions and projections that appear in Definition 1 can be chosen to depend analytically on* λ *for* λ *close to* λ_* .

We may wish to replace the condition $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ that appears in Definition 1 by the weaker condition $\kappa^{\mathrm{s}} < \kappa^{\mathrm{u}}$. Using a transformation for an appropriate η, we see that all the results mentioned above are also true under this weaker condition, i.e. for arbitrary spectral gaps.

Spectra and Fredholm Properties

We consider the family of operators

$$\mathcal{T}(\lambda) \; : \qquad \mathcal{D} \longrightarrow \mathcal{H}, \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u \qquad (8)$$

with parameter λ .

Spectra and Fredholm Properties

We consider the family of operators

$$\mathcal{T}(\lambda) : \qquad \mathcal{D} \longrightarrow \mathcal{H}, \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u \qquad (8)$$

with parameter λ .

Recall that an operator $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is said to be a Fredholm operator if $\mathrm{R}(\mathcal{L})$ is closed in $\mathcal{Y}$, and the dimension of $\mathrm{N}(\mathcal{L})$ and the codimension of $\mathrm{R}(\mathcal{L})$ are both finite.
The difference $\dim \mathrm{N}(\mathcal{L}) - \operatorname{codim} \mathrm{R}(\mathcal{L})$ is called the Fredholm index of $\mathcal{L}$.

Definition 2 (**Spectrum**) *We say that* λ *is in the spectrum* Σ *of* $\mathcal{T}$ *if* $\mathcal{T}(\lambda)$ *is not invertible, i.e. if the inverse operator does not exist or is not bounded. We say that* $\lambda \in \Sigma$ *is in the point spectrum* Σ_{pt} *of* $\mathcal{T}$ *or, alternatively, that* $\lambda \in \Sigma$ *is an eigenvalue of* $\mathcal{T}$ *if* $\mathcal{T}(\lambda)$ *is a Fredholm operator with index zero. The complement* $\Sigma \setminus \Sigma_{\mathrm{pt}} =: \Sigma_{\mathrm{ess}}$ *is called the essential spectrum. The complement of* Σ *in* $\mathbb{C}$ *is the resolvent set of* $\mathcal{T}$.

Definition 2 **(Spectrum)** *We say that λ is in the spectrum Σ of $\mathcal{T}$ if $\mathcal{T}(\lambda)$ is not invertible, i.e. if the inverse operator does not exist or is not bounded. We say that $\lambda \in \Sigma$ is in the point spectrum Σ_{pt} of $\mathcal{T}$ or, alternatively, that $\lambda \in \Sigma$ is an eigenvalue of $\mathcal{T}$ if $\mathcal{T}(\lambda)$ is a Fredholm operator with index zero. The complement $\Sigma \setminus \Sigma_{\mathrm{pt}} =: \Sigma_{\mathrm{ess}}$ is called the essential spectrum. The complement of Σ in $\mathbb{C}$ is the resolvent set of $\mathcal{T}$.*

Remark 2 *The point spectrum is often defined as the set of all isolated eigenvalues with finite multiplicity, i.e. as the set $\tilde{\Sigma}_{\mathrm{pt}}$ of those λ for which $\mathcal{T}(\lambda)$ is Fredholm with index zero, the null space of $\mathcal{T}(\lambda)$ is non-trivial, and $\mathcal{T}(\tilde{\lambda})$ is invertible for all $\tilde{\lambda}$ in a small neighbourhood of λ (except, of course, for $\tilde{\lambda} = \lambda$).*

Theorem 2 *Fix $\lambda \in \mathbb{C}$. The following statements are true.*

- *λ is in the resolvent set of $\mathcal{T}$ if, and only if,* (4) *has an exponential dichotomy on $\mathbb{R}$.*

Theorem 2 *Fix $\lambda \in \mathbb{C}$. The following statements are true.*

- λ *is in the resolvent set of* $\mathcal{T}$ *if, and only if,* (4) *has an exponential dichotomy on* $\mathbb{R}$.
- λ *is in the point spectrum* Σ_{pt} *of* $\mathcal{T}$ *if, and only if,* (4) *has exponential dichotomies on* $\mathbb{R}^+$ *and on* $\mathbb{R}^-$ *with the same Morse index,* $i_+(\lambda) = i_-(\lambda)$, *and* $\dim \mathrm{N}(\mathcal{T}(\lambda)) > 0$. *In this case, denote by* $P_\pm(\xi;\lambda)$ *the projections of the exponential dichotomies of* (4) *on* $\mathbb{R}^\pm$, *then the spaces* $\mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda))$ *and* $\mathrm{N}(\mathcal{T}(\lambda))$ *are isomorphic via* $u(0) \mapsto u(\cdot)$.

Theorem 2 *Fix $\lambda \in \mathbb{C}$. The following statements are true.*

- *λ is in the resolvent set of $\mathcal{T}$ if, and only if,* (4) *has an exponential dichotomy on $\mathbb{R}$.*
- *λ is in the point spectrum Σ_{pt} of $\mathcal{T}$ if, and only if,* (4) *has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with the same Morse index, $i_+(\lambda) = i_-(\lambda)$, and $\dim \mathrm{N}(\mathcal{T}(\lambda)) > 0$. In this case, denote by $P_\pm(\xi;\lambda)$ the projections of the exponential dichotomies of* (4) *on $\mathbb{R}^\pm$, then the spaces $\mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda))$ and $\mathrm{N}(\mathcal{T}(\lambda))$ are isomorphic via $u(0) \mapsto u(\cdot)$.*
- *λ is in the essential spectrum Σ_{ess} if* (4) *either does not have exponential dichotomies on $\mathbb{R}^+$ or on $\mathbb{R}^-$, or else if it does, but the Morse indices on $\mathbb{R}^+$ and on $\mathbb{R}^-$ differ.*

Theorem 2 *Fix $\lambda \in \mathbb{C}$. The following statements are true.*

- *λ is in the resolvent set of $\mathcal{T}$ if, and only if,* (4) *has an exponential dichotomy on $\mathbb{R}$.*
- *λ is in the point spectrum Σ_{pt} of $\mathcal{T}$ if, and only if,* (4) *has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with the same Morse index, $i_+(\lambda) = i_-(\lambda)$, and $\dim \mathrm{N}(\mathcal{T}(\lambda)) > 0$. In this case, denote by $P_\pm(\xi;\lambda)$ the projections of the exponential dichotomies of* (4) *on $\mathbb{R}^\pm$, then the spaces $\mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda))$ and $\mathrm{N}(\mathcal{T}(\lambda))$ are isomorphic via $u(0) \mapsto u(\cdot)$.*
- *λ is in the essential spectrum Σ_{ess} if* (4) *either does not have exponential dichotomies on $\mathbb{R}^+$ or on $\mathbb{R}^-$, or else if it does, but the Morse indices on $\mathbb{R}^+$ and on $\mathbb{R}^-$ differ.*

Remark 3 *To summarize the relation between Fredholm properties of $\mathcal{T}$ and exponential dichotomies of* (4) *, we remark that $\mathcal{T}$ is Fredholm if, and only if,* (4) *has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$. The Fredholm index of $\mathcal{T}$ is then equal to the difference $i_-(\lambda) - i_+(\lambda)$ of the Morse indices of the dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. If $\mathcal{T}(\lambda)$ is not Fredholm, then typically the range $\mathrm{R}(\mathcal{T}(\lambda))$ of $\mathcal{T}(\lambda)$ is not closed in $\mathcal{H}$.*

Remark 4 *Suppose that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;\lambda)u \tag{9}$$

has an exponential dichotomy on I *with projections* $P(\xi;\lambda)$ *and evolutions* $\Phi^{\mathrm{s}}(\xi,\zeta;\lambda)$ *and* $\Phi^{\mathrm{u}}(\xi,\zeta;\lambda)$*, then the equation*

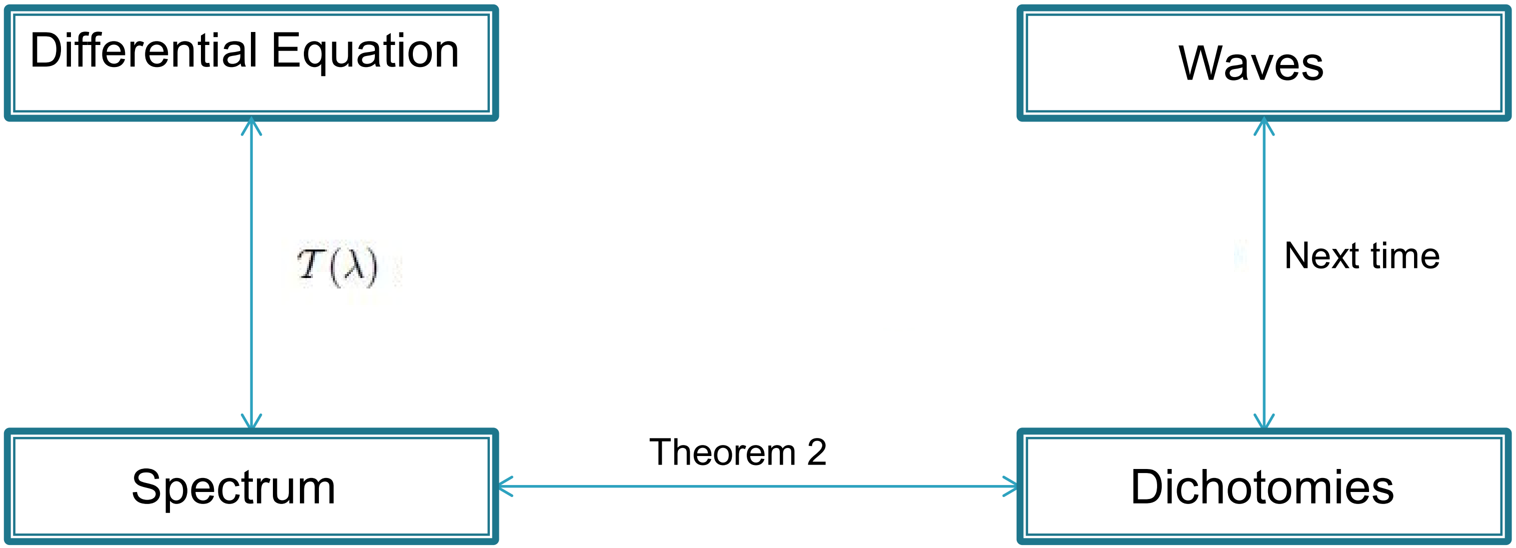
$$\frac{\mathrm{d}}{\mathrm{d}\xi}v = -A(\xi;\lambda)^{*}v \tag{10}$$

also has an exponential dichotomy on I *with projections* $\tilde{P}(\xi;\lambda)$ *and evolutions* $\tilde{\Phi}^{\mathrm{s}}(\xi,\zeta;\lambda)$ *and* $\tilde{\Phi}^{\mathrm{u}}(\xi,\zeta;\lambda)$*. The projections and evolutions of* (9) *and* (10) *are related via*

$$\tilde{P}(\xi;\lambda) = \mathrm{id} - P(\xi;\lambda)^{*}, \qquad \tilde{\Phi}^{\mathrm{s}}(\xi,\zeta;\lambda) = \Phi^{\mathrm{u}}(\zeta,\xi;\lambda)^{*}, \qquad \tilde{\Phi}^{\mathrm{u}}(\xi,\zeta;\lambda) = \Phi^{\mathrm{s}}(\zeta,\xi;\lambda)^{*}.$$

This is a consequence of Definition 1 .

Summary



Differential Equation
Waves
$T(\lambda)$
Next time
Theorem 2
Spectrum
Dichotomies