Stability of Travelling Waves

Waves: Spectrum & Evans Function
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Last Lecture

Differential Equation

\[ T(\lambda) \]

Spectrum

Waves

definition of waves

Dichotomies

Theorem 1
Differential Equation

\[ U_t = A(\partial_x)U + N(U) \]
Differential Equation

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Wave solution

\[ Q(x - ct) \]
Differential Equation

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Linearized Equation

Wave solution

\[ Q(x - ct) \]

\[ \xi = x - ct \]
Differential Equation
\[ U_t = A(\partial_x)U + N(U) \]

Wave solution
\[ Q(x - ct) \]

Linearized Equation
\[ \xi = x - ct \]

Eigenvalue Equation
\[ \frac{d}{d\xi} u = A(\xi; \lambda)u \]

U = e^{\lambda t}u
Differential Equation

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Wave solution

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Linearized Equation

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Spectrum
\[ T(\lambda)u = \frac{du}{d\xi} - A(\cdot; \lambda)u \]
Theorem 1
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Spectrum

Resolvent
Theorem 1

Point Spectrum

Essential Spectrum

Resolvent
Theorem 1

Point Spectrum

Essential Spectrum

Resolvent

Exponentional dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with $i_+(\lambda) = i_-(\lambda)$
Theorem 1

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- Point Spectrum
- Essential Spectrum
- Resolvent
- Exponentional dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with $i_+(\lambda) = i_-(\lambda)$
- Other function
- Exponentional dichotomy on $\mathbb{R}$
Point Spectrum

Essential Spectrum

Resolvent

Instable wave if totally in right half plane
Waves

Most common types

__________________________ Homogeneous rest states
Waves

Most common types

Homogeneous rest states

Front and back
Waves

Most common types

Homogeneous rest states

Front and back

Pulse
Waves

Most common types

- Homogeneous rest states
- Front and back
- Pulse
- Periodic wave train
Homogeneous rest states

The matrix $A(\xi; \lambda)$ is independent of position and has eigenvalues $\mu$. 
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point spectrum is empty.
Fronts and Backs

The wave has two asymptotic rest states 

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Assume that for \( \xi \) large there exist matrices of \( A(\xi; \lambda) \).

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in resolvent set iff \( A_{\pm}(\lambda) \) is hyperbolic, \( i_+(\lambda) = i_-(\lambda) \) and \( N(P_-(0; \lambda)) \oplus R(P_+(0; \lambda)) = \mathbb{C}^n \) hold.
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\[ \lambda \text{ in point spectrum iff } A_\pm(\lambda) \text{ is hyperbolic, } i_+(\lambda) = i_- (\lambda) \text{ and } \]

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\[ \lambda \text{ in essential spectrum iff none of the above.} \]
Pulses

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Periodic Wave Train

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R(\lambda) &\in \mathbb{C}^{n \times n}
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- Point spectrum is empty.
Instability types

Absolute instability
Instability types

Absolute instability

Convective instability
The Evans function

We assume a Morse index $k$. Then we obtain ordered bases
\[ u_{k+1}(\lambda), \ldots, u_n(\lambda) \text{ and } u_1(\lambda), \ldots, u_k(\lambda) \]
of spaces $R(P_+(0; \lambda))$ and $N(P_-(0; \lambda))$. 
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The Evans function $D(\lambda)$ is designed to locate non-trivial intersections of $R(P_+(0; \lambda))$ and $N(P_-(0; \lambda))$. 

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**Definition 4.1 (The Evans function)** The Evans function is defined by

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$$D(\lambda) = N(P_-(0; \lambda)) \cap R(P_+(0; \lambda))$$
Theorem 4.1 The Evans function $D(\lambda)$ is analytic in $\lambda \in \Omega$ and has the following properties.

- $D(\lambda) \in \mathbb{R}$ whenever $\lambda \in \mathbb{R} \cap \Omega$.
- $D(\lambda) = 0$ if, and only if, $\lambda$ is an eigenvalue of $T$.
- The order of $\lambda_*$ as a zero of the Evans function $D(\lambda)$ is equal to the algebraic multiplicity of $\lambda_*$ as an eigenvalue of $T$. 
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Diagram:

- Differential Equation
  - $T(\lambda)$
- Spectrum
- Waves
  - definition of waves
- Dichotomies
  - Theorem 1
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Differential Equation

\[ D(\lambda) \]

\[ T(\lambda) \]

Spectrum

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definition of waves

Theorem 1

Dichotomies