On the Top of a Function.
Maximum Principle and Sub-/Supersolutions

Dirk van Kekem

April 18, 2012
1 Differential Operators

2 The Maximum Principle
   - The Weak Maximum Principle
   - The Strong Maximum Principle
   - Application to Boundary-Value Problems

3 Eigenvalues and Solutions
   - Principal Eigenvalue
   - Sub- and Supersolutions

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   - The Weak Maximum Principle
   - The Strong Maximum Principle
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   - Principal Eigenvalue
   - Sub- and Supersolutions

4. **Outlook**
Study boundary value problems: bounded, open $U \subset \mathbb{R}^n$.

To find: $u : \overline{U} \to \mathbb{R}$.

Let $f : U \to \mathbb{R}, g : \partial U \to \mathbb{R}$ given functions.

\[
\begin{aligned}
Lu &= f(x) \text{ in } U \\
u &= g(x) \text{ on } \partial U,
\end{aligned}
\tag{1}
\]

where $L$ a second-order partial differential operator, given by

\[
Lu = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u + \sum_{i=1}^{n} b_{i}(x) \partial_{i} u + c(x) u.
\tag{2}
\]
Elliptic Operators

Definition (Elliptic Operator)

$L$ is an \textit{(uniformly) elliptic operator} if there exists $\theta > 0$ such that for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$,

$$\theta |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j. \quad (3)$$

$L$ is \textit{pointwise elliptic} if $\theta$ depends on $x \in U$.

$L$ is \textit{elliptic degenerate} if $0 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j$, and there exists a fixed unit vector $\zeta$ such that for all $x \in U$,

$$\theta \leq \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i \zeta_j. \quad (4)$$
Definition (Elliptic Operator)

$L$ is an *(uniformly) elliptic operator* if there exists $\theta > 0$ such that for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$,

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$$\theta \leq \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j. \quad (4)$$
Definition (Parabolic Operator)

Let $Q := (0, T) \times U$ for some $T > 0$. A parabolic operator is operator of the form

$$P := \partial_t - \sum_{i,j=1}^{n} a_{ij}(t, x) \partial_{ij} - \sum_{i=1}^{n} b_{i}(t, x) \partial_{i} - c(t, x),$$  \hspace{1cm} (5)$$

where coefficients satisfy ellipticity conditions.
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where coefficients satisfy ellipticity conditions.

Can write: $P = \partial_t - L$. 
Example (Elliptic Operators)

- Laplace operator: \( \Delta u = \sum_{i=1}^{n} u_{x_{i}x_{i}} = 0; \)
- Helmholtz equation: \( \Delta u + \lambda u = 0. \)
Examples

Example (Elliptic Operators)

- Laplace operator: $\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0$;
- Helmholtz equation: $\Delta u + \lambda u = 0$.

Example (Parabolic Operators)

- Heat operator: $u_t - \Delta u = 0$;
- Kolmogorov’s equation: $u_t - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} = 0$;
- Scalar reaction-diffusion equation: $u_t - \Delta u = f(u)$.
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2. The Maximum Principle
   - The Weak Maximum Principle
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4. Outlook
The Weak Maximum Principle

Assumptions:
From now on: bounded, open $U \subset \mathbb{R}^n$ with boundary smooth enough.
Furthermore: $u \in C^2(U) \cap C(\overline{U})$. 

Theorem (Weak Maximum Principle)
Let $L$ elliptic (degenerate) operator with $Lu \geq 0$ in $U$.

1. If $c(x) \equiv 0$ in $U$ then $\max_{U} u = \max_{\partial U} u$.
2. If $c(x) \leq 0$ in $U$ and $\max_{U} u \geq 0$, then $\max_{U} u = \max_{\partial U} u$.

Weaker form:
If not assumed $\max_{U} u \geq 0$, then $\max_{U} u \leq \max_{\partial U} u +$. 

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The Weak Maximum Principle

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All results also for $Lu \leq 0$: gives “$\min_{\overline{U}} u$”.
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The Weak Maximum Principle

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From now on: bounded, open $U \subset \mathbb{R}^n$ with boundary smooth enough. Furthermore: $u \in C^2(U) \cap C(\bar{U})$.

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Weaker form:

2. If not assumed $\max_{\bar{U}} u \geq 0$, then $\max_{\bar{U}} u \leq \max_{\partial U} u^+$. 
Proof (1).

Consider first $Lu > 0$ in $U$. Then maximum on boundary: $x_0 \in \partial U$, with

$$Du(x_0) = 0; \quad D^2 u(x_0) \leq 0.$$  \hfill (6)
Proof (1).

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$x_0 \notin U$: for nonnegative, symmetric matrices $(\alpha_{ij}), (\beta_{ij})$, \[ \sum_{i,j=1}^{n} \alpha_{ij} \beta_{ij} \geq 0. \]
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So, at $x_0$:

$$Lu(x_0) = \sum_{i,j=1}^{n} a_{ij} \partial_{ij} u(x_0) + \sum_{i=1}^{n} b_i \partial_i u(x_0) = \sum_{i,j=1}^{n} a_{ij}(x_0) \partial_{ij} u(x_0) \leq 0, \quad (7)$$

contradiction.
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\]

contradiction.

General: define \( u_\varepsilon := u(x) + \varepsilon e^{\lambda x_1} \), \( \varepsilon > 0, \lambda > 0 \) sufficiently large. This function satisfies:

\[
Lu_\varepsilon > 0 \text{ in } U \quad \Rightarrow \quad \max_{\overline{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon. \tag{8}
\]
Proof (1).

Consider first \( Lu > 0 \) in \( U \). Then maximum on boundary: \( x_0 \in \partial U \), with

\[
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General: define \( u_\varepsilon := u(x) + \varepsilon e^{\lambda x_1} \), \( \varepsilon > 0 \), \( \lambda > 0 \) sufficiently large. This function satisfies:

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Lu_\varepsilon > 0 \text{ in } U \quad \Rightarrow \quad \max_{\overline{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon. \tag{8}
\]

Letting \( \varepsilon \to 0 \) gives the result.
Proof (2).

Let \( U^+ := \{ u > 0 \} \subset U \), then

\[
Mu = Lu - c(x)u \geq 0 \text{ on } U^+
\]

\[
u = 0 \text{ on } \partial U^+.
\]

(9)
Proof (2).

Let $U^+ := \{ u > 0 \} \subset U$, then

\[ Mu = Lu - c(x)u \geq 0 \text{ on } U^+ \]
\[ u = 0 \text{ on } \partial U^+. \]  \hspace{1cm} (9)

Hence, by part (1), if $U^+ \neq \emptyset$:

\[ 0 \leq \max_{\overline{U}} u = \max_{\overline{U}^+} u = \max_{\partial U^+} u = \max_{\partial U^+ \cap \partial U} u = \max_{\partial U} u. \]  \hspace{1cm} (10)
Proof (2).

Let $U^+ := \{u > 0\} \subset U$, then

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\quad u = 0 \text{ on } \partial U^+.$$ (9)

Hence, by part (1), if $U^+ \neq \emptyset$:

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Otherwise: $u \leq 0$ everywhere.
Corollary

Let $L$ an elliptic (degenerate) operator with $c(x) \leq 0$ in $U$. If $Lu \geq 0$ in $U$ and $u \leq 0$ on $\partial U$, then $u \leq 0$ in $U$. 
Parabolic boundary of $Q$: $\partial_p Q := \{0\} \times \overline{U} \cup \{[0, T] \times \partial U\}$.

**Theorem (Weak Maximum Principle for Parabolic Operator)**

Let $P = \partial_t - L$ a parabolic degenerate operator, $u \in C^1$ wrt. $t$, such that $Pu \leq 0$ in $U$.

1. If $c(t, x) \equiv 0$, or
2. if $c(t, x) \leq 0$ and $\max_Q u \geq 0$,

then: $\max_Q u = \max_{\partial_p Q} u$. 
Parabolic boundary of \( Q \): \( \partial_p Q := \{0\} \times \overline{U} \cup \{0, T\} \times \partial U \).

**Theorem (Weak Maximum Principle for Parabolic Operator)**

Let \( P = \partial_t - L \) a parabolic degenerate operator, \( u \in C^1 \) wrt. \( t \), such that \( Pu \leq 0 \) in \( U \).

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2. if \( c(t, x) \leq 0 \) and \( \max_Q u \geq 0 \),

then: \( \max_Q u = \max_{\partial_p Q} u \).

**Proof.**

The proof goes like the elliptic case.
The Strong Maximum Principle

Theorem (Strong Maximum Principle)

Let $L$ be elliptic operator, $U$ connected, $u$ such that $Lu \geq 0$ in $U$.

1. If $c \equiv 0$ and $\max_{\overline{U}} u = u(x_0)$ at interior point $x_0 \in U$, then $u$ constant in $U$.

2. If $c(x) \leq 0$ and $u(x_0) = \max_{U} u \geq 0$, then $u$ constant in $U$.

The proof uses Hopf’s Lemma.
The Strong Maximum Principle

Theorem (Strong Maximum Principle)

Let $L$ be elliptic operator, $U$ connected, $u$ such that $Lu \geq 0$ in $U$.

1. If $c \equiv 0$ and $\max_{\overline{U}} u = u(x_0)$ at interior point $x_0 \in U$, then $u$ constant in $U$.

2. If $c(x) \leq 0$ and $u(x_0) = \max_U u \geq 0$, then $u$ constant in $U$.

The proof uses *Hopf’s Lemma*. 
Hopf’s Lemma

**Lemma (Hopf)**

Let $L$ and $u$ as before. Suppose there exists $p \in \partial U$ such that $u(p) > u(x)$ for all $x \in U$.

1. If $c \equiv 0$ in $U$, then
   \[
   \frac{\partial u}{\partial \xi}(p) > 0,
   \]
   where $\xi$ the outer unit normal at $p$.

2. If $c \leq 0$ in $U$ and $u(p) \geq 0$, then same result holds.
Dirichlet problem: let $f : U \rightarrow \mathbb{R}, g : \partial U \rightarrow \mathbb{R}$ given functions.

**Theorem**

When it exists, the solution of

$$\begin{cases} 
Lu = f(x) \text{ in } U \\
u = g(x) \text{ on } \partial U,
\end{cases}$$

(12)

is unique.
Dirichlet problem: let \( f : U \rightarrow \mathbb{R}, \ g : \partial U \rightarrow \mathbb{R} \) given functions.

**Theorem**

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Lu &= f(x) \text{ in } U \\
\ u &= g(x) \text{ on } \partial U,
\end{aligned}
\]

is unique.

**Proof.**

Difference \( w = \nu - u \) of two solutions \( u, \nu \) satisfies homogeneous problem. From the Corollary, it follows that \( w \leq 0 \) and \( w \geq 0 \), hence \( w \equiv 0 \).
Dirichlet problem: let $f : U \rightarrow \mathbb{R}, g : \partial U \rightarrow \mathbb{R}$ given functions.

Theorem

When it exists, the solution of

$$
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  Lu = f(x) \text{ in } U \\
  u = g(x) \text{ on } \partial U,
\end{cases}
$$

is unique.

Proof.

Difference $w = v - u$ of two solutions $u, v$ satisfies homogeneous problem. From the Corollary, it follows that $w \leq 0$ and $w \geq 0$, hence $w \equiv 0$.

Similar results for other boundary value problems.
Progression

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3. Eigenvalues and Solutions
   - Principal Eigenvalue
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4. Outlook
Definition

Function $u : U \rightarrow \mathbb{R}$ is of class $C^{k, \gamma}$, $0 < \gamma < 1$, if the norm

$$
\|u\|_{C^{k, \gamma}(U)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0, \gamma}(\overline{U})}
$$

$$
:= \sum_{|\alpha| \leq k} \sup_{x \in U} |D^\alpha u(x)| + \sum_{|\alpha| = k} \sup_{x, y \in U} \left( \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma} \right)
$$

(13)

is finite.

These functions constitute the Hölder space $C^{k, \gamma}(\overline{U})$, which is Banach.
Definition

Let $U$ be domain with $\partial U$ of class $C^{2,\gamma}$; $L$ elliptic operator with coefficients of class $C^{0,\gamma}(\overline{U})$. Suppose $\varphi_1 \geq 0$ is eigenfunction of $-L$, which satisfies

\[
\begin{cases}
\varphi_1 > 0 \text{ in } U \\
\frac{\partial \varphi_1}{\partial \xi} < 0 \text{ on } \partial U.
\end{cases}
\]

(14)
Eigenvalue $\lambda_1$ corresponding to $\varphi_1$ is simple and has $\lambda_1 \leq \Re(\lambda)$. Eigenvalue $\lambda_1$ is called principal eigenvalue and eigenfunction $\varphi_1$ principal eigenfunction.
Principal Eigenvalue

Definition

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(15)
Definition

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Eigenvalue $\lambda_1$ corresponding to $\varphi_1$ is simple and has

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\lambda_1 \leq \Re(\lambda).
$$

Eigenvalue $\lambda_1$ is called principal eigenvalue and eigenfunction $\varphi_1$ principal eigenfunction.
Existence and Uniqueness

Theorem

This eigenvalue $\lambda_1$ exists and is unique.
Theorem
This eigenvalue $\lambda_1$ exists and is unique.

Proof.
The proof uses Krein-Rutman theory, by taking an order on space $C^1_0(\overline{U})$. 
Sub- and Supersolutions

Want to find $C^2$-solution of

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Lu + f(x, u) &= 0 \text{ in } U \\
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\] (16)
Sub- and Supersolutions

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\end{aligned}
\]  

(16)

Definition (Sub-/Supersolution)

A sub\-solution is function $u \in C^2(U)$ satisfying

\[
\begin{aligned}
Lu + f(x, u) &\geq 0 \text{ in } U \\
u &\leq 0 \text{ on } \partial U.
\end{aligned}
\]  

(17)

Similarly, a sup\-solution is function $\bar{u} \in C^2(U)$ satisfying

\[
\begin{aligned}
L\bar{u} + f(x, \bar{u}) &\leq 0 \text{ in } U \\
\bar{u} &\geq 0 \text{ on } \partial U.
\end{aligned}
\]  

(18)
Theorem

Let $U$ of class $C^{2,\gamma}$, $L$ elliptic operator with coefficients of class $C^{0,\gamma}$ and $f : \overline{U} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

For any $r > 0$, there exists $C(r) > 0$ such that for all $x, y \in \overline{U}, s, t \in [-r, r]$

$$|f(x, s) - f(y, t)| \leq C(|x - y|^\gamma + |s - t|).$$  \hspace{1cm} (19)
Theorem

Let $U$ of class $C^{2,\gamma}$, $L$ elliptic operator with coefficients of class $C^{0,\gamma}$ and $f : \overline{U} \times \mathbb{R} \to \mathbb{R}$ satisfying:

For any $r > 0$, there exists $C(r) > 0$ such that for all $x, y \in \overline{U}$, $s, t \in [-r, r]$

$$|f(x, s) - f(y, t)| \leq C(|x - y|^\gamma + |s - t|).$$

(19)

Assume there exists a subsolution $u$ and a supersolution $\overline{u}$, both $C^{0,\gamma}$, such that $u \leq \overline{u}$. Then there exist at least one solution $u$ with $u \leq u \leq \overline{u}$. Moreover, there is a minimal and a maximal one.
Theorem

Let $U$ of class $C^{2,\gamma}$, $L$ elliptic operator with coefficients of class $C^{0,\gamma}$ and $f : \overline{U} \times \mathbb{R} \to \mathbb{R}$ satisfying:

For any $r > 0$, there exists $C(r) > 0$ such that for all $x, y \in \overline{U}$, $s, t \in [-r, r]$

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Assume there exists a subsolution $u$ and a supersolution $\overline{u}$, both $C^{0,\gamma}$, such that $u \leq \overline{u}$. Then there exist at least one solution $u$ with $u \leq u \leq \overline{u}$. Moreover, there is a minimal and a maximal one.

Can generalize to Sobolev spaces. Then $f(x, s)$ has Carathéodory conditions:

$$\begin{cases} x \to f(x, s) \text{ is measurable in } x, \text{ for all } s \in \mathbb{R}, \\ s \to f(x, s) \text{ is continuous in } s, \text{ for a.e. } x \in U. \end{cases}$$ \hspace{1cm} (20)
Proof.

Consider sequences of functions \((v_n)\) and \((w_n)\), solutions for \(L - C + f(x, \cdot) + Cs\) and satisfying:

\[-r \leq u = v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = \bar{u} \leq r.\]

(21)
Proof.

Consider sequences of functions \((v_n)\) and \((w_n)\), solutions for \(L - C + f(x, \cdot) + Cs\) and satisfying:

\[-r \leq u = v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = \bar{u} \leq r.\]

\((v_n)\) and \((w_n)\) converge to solutions \(v, w \in C^2, \gamma\)
Proof.

Consider sequences of functions \((v_n)\) and \((w_n)\), solutions for 
\[ L - C + f(x, \cdot) + Cs \] and satisfying:
\[-r \leq u = v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = \overline{u} \leq r.\]

(21)

\((v_n)\) and \((w_n)\) converge to solutions \(v, w \in C^{2,\gamma}\)

If \(u \in C^2\) is solution with \(u \leq u \leq \overline{u}\), then \(v \leq u \leq w\).
What happens if we start with some sub-/supersolution?
What happens if we start with some sub-/supersolution?

- If subsolution $u$ blows up, then solution $u$ blows up;
- If subsolution $u$ blows up in finite time, then solution $u$ blows up in finite time;
- If $\bar{u}$ is global (in time) supersolution above $u$, then $u$ global.
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Further reading

- Evans, Lawrence C.
  *Partial Differential Equations*

- Berestycki, Henri and Hamel, François
  *Chapter 1: The Maximum Principle*
  Yet unpublished.

Next time: more dynamics
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To be continued