Application of semigroup theory to reaction-diffusion equations

Martino Pitruzzella

16 May 2012
Outline

Aim of the talk, introduction and motivation
Summary of semigroup theory
  - Definitions
  - Example
  - Theorem
  - Hille Yoshida theorem
  - More definitions and theorems
How the theory is applied
  - Laplace operator
  - Abstract evolution equation
  - Reaction-diffusion equations
Principle of linearized stability
Example: Turing instability on interval
The aim of this seminar talk is to show how semigroup theory can be used to study evolution equations of the kind:
The aim of this seminar talk is to show how semigroup theory can be used to study evolution equations of the kind:

$$\dot{x} = Ax + N(x), \ x(0) = x_0, \ x_0 \in X$$
The aim of this seminar talk is to show how semigroup theory can be used to study evolution equations of the kind:
\[ \dot{x} = Ax + N(x), \quad x(0) = x_0, \quad x_0 \in X \]
where $X$ is a Banach space, $A : D(A) \subset X \to X$ is a linear operator and $N : X \to X$ is (non-linear) and smooth.
The aim of this seminar talk is to show how semigroup theory can be used to study evolution equations of the kind:

\[ \dot{x} = Ax + N(x) , \quad x(0) = x_0 , \quad x_0 \in X \]

where \( X \) is a Banach space, \( A : D(A) \subset X \rightarrow X \) is a linear operator and \( N : X \rightarrow X \) is (non-linear) and smooth. In particular we are interested in reaction-diffusion systems of the form:

\[ \dot{u} = D\Delta u + Cu + f(u) \]
The aim of this seminar talk is to show how semigroup theory can be used to study evolution equations of the kind:
\[ \dot{x} = Ax + N(x) , \ x(0) = x_0 , \ x_0 \in X \]
where \( X \) is a Banach space, \( A : D(A) \subset X \to X \) is a linear operator and \( N : X \to X \) is (non-linear) and smooth.
In particular we are interested in reaction-diffusion systems of the form: \( \dot{u} = D \Delta u + Cu + f(u) \)
on a bounded domain \( \Omega \subset \mathbb{R}^m (m \leq 3) \) with Dirichlet or Neumann conditions on its piecewise smooth boundary \( \partial \Omega \) where \( u = (u_1, .., u_n)^T, D = diag(d_1, .., d_n) \) is a diagonal matrix, \( C = (c_{ij}) \) and \( f = (f_1, .., f_n)^T. \)
The aim of this seminar talk is to show how semigroup theory can be used to study evolution equations of the kind:
\[ \dot{x} = Ax + N(x) , \ x(0) = x_0 , \ x_0 \in X \]
where \( X \) is a Banach space, \( A : D(A) \subset X \to X \) is a linear operator and \( N : X \to X \) is (non-linear) and smooth.
In particular we are interested in reaction-diffusion systems of the form:
\[ \dot{u} = D\Delta u + Cu + f(u) \]
on a bounded domain \( \Omega \subset \mathbb{R}^m (m \leq 3) \) with Dirichlet or Neumann conditions on its piecewise smooth boundary \( \partial \Omega \) where
\[ u = (u_1, .., u_n)^T, \ D = \text{diag}(d_1, .., d_n) \text{ is a diagonal matrix,} \]
\[ C = (c_{ij}) \text{ and } f = (f_1, .., f_n)^T. \]
Semigroup theory is a way to see evolution equations of the form:
\[ \frac{d}{dt} u(t) = R(u(t)) \] where \( R \) is an operator, as ODEs on a Banach function space.
A semigroup is a set \((S, \ast)\) with a binary operation \(\ast\) which is associative: \(\forall x, y, z \in S, (x \ast y) \ast z = x \ast (y \ast z)\)
A semigroup is a set \((S, \ast)\) with a binary operation \(\ast\) which is associative: \(\forall x, y, z \in S, (x \ast y) \ast z = x \ast (y \ast z)\)

We are interested in semigroups of bounded linear operators on a Banach space \(X\). A one-parameter family
\[
T = T(t) = \{ T(t) \mid t \in \mathbb{R}_+ \}, \quad T(t) : X \to X
\]
A semigroup is a set \((S, *)\) with a binary operation \(*\) which is associative: \(\forall x, y, z \in S, (x * y) * z = x * (y * z)\)

We are interested in semigroups of bounded linear operators on a Banach space \(X\). A one-parameter family \(T = T(t) = \{ T(t) \mid t \in \mathbb{R}_+ \}, T(t) : X \to X\) satisfying:

\(\blacktriangleright\) \(T(0) = I\)
Definitions

A semigroup is a set \((S, \ast)\) with a binary operation \(\ast\) which is associative: \(\forall x, y, z \in S, (x \ast y) \ast z = x \ast (y \ast z)\)

We are interested in semigroups of bounded linear operators on a Banach space \(X\). A one-parameter family
\[ T = T(t) = \{ T(t) \mid t \in \mathbb{R}_+ \} , \quad T(t) : X \to X \]
satisfying:
\[
\begin{align*}
&\text{\(T(0) = I\)} \\
&\text{\(T(t + s) = T(t)T(s)\), \(\forall t, s \in \mathbb{R}_+\)}
\end{align*}
\]
A semigroup is a set \((S, \ast)\) with a binary operation \(\ast\) which is associative: 
\[\forall x, y, z \in S, (x \ast y) \ast z = x \ast (y \ast z)\]

We are interested in semigroups of bounded linear operators on a Banach space \(X\). A one-parameter family  
\[T = T(t) = \{ T(t) | t \in \mathbb{R}_+ \}, T(t) : X \to X\]
satisfying:

- \(T(0) = I\)
- \(T(t + s) = T(t)T(s), \forall t, s \in \mathbb{R}_+\)
- \(T(t)w \to x\) as \(t \to 0^+, \forall x \in X\)
Definitions

A semigroup is a set \((S, \ast)\) with a binary operation \(\ast\) which is associative: \(\forall x, y, z \in S, (x \ast y) \ast z = x \ast (y \ast z)\)

We are interested in semigroups of bounded linear operators on a Banach space \(X\). A one-parameter family

\[
T = T(t) = \{ T(t) \mid t \in \mathbb{R}_+ \}, \ T(t) : X \to X
\]

satisfying:

\[
\begin{align*}
&\quad T(0) = I \\
&\quad T(t + s) = T(t)T(s), \forall t, s \in \mathbb{R}_+ \\
&\quad T(t)w \to x \text{ as } t \to 0^+, \forall x \in X
\end{align*}
\]

Is called a strongly continous semigroup or \(C^0\) semigroup.
Definitions

If instead of the last condition we had:
If instead of the last condition we had:

\[ \lim_{t \to 0^+} T(t) = I \]

then \( T \) is called uniformly continuous.
Definitions

If instead of the last condition we had:

\[ \lim_{t \to 0^+} T(t) = I \]

then \( T \) is called uniformly continuous.

Moreover if:

\[ \| T(t) \| \leq 1, \forall t \geq 0, \] \( T \) is called semigroup of contractions.
Definition of semigroup generator

The infinitesimal generator $C$ of the semigroup $T(t)$ is defined as:

$$Cw = \lim_{t \to 0^+} T(t)w - w$$

It is defined on its domain $D(C) \subseteq X$, the set where the limit exists. It is proven that $D(C)$ is dense in $X$ and $C$ is a closed operator.
Definition of semigroup generator

The infinitesimal generator $C$ of the semigroup $T(t)$ is defined as:

$$Cw = \lim_{t \to 0^+} \frac{T(t)x - x}{t}$$
The infinitesimal generator $C$ of the semigroup $T(t)$ is defined as:

$$CW = \lim_{t \to 0^+} \frac{T(t)x - x}{t}$$

It is defined on its domain $D(C) \subseteq X$, the set where the limit exists.
Definition of semigroup generator

The infinitesimal generator $C$ of the semigroup $T(t)$ is defined as:

$$Cw = \lim_{t \to 0^+} \frac{T(t)x - x}{t}$$

It is defined on its domain $D(C) \subseteq X$, the set where the limit exists.

It is proven that $D(C)$ is dense in $X$ and $C$ is a closed operator.
Examples

- If $X$ is the Banach space of bounded uniformly continuous functions on $\mathbb{R}_+$ with supremum norm.
Examples

- If $X$ is the Banach space of bounded uniformly continuous functions on $\mathbb{R}_+$ with supremum norm. Define $(T(t)f)(\theta) = f(\theta + t), f \in X, \theta \geq 0, t \geq 0$.
Examples

- If $X$ is the Banach space of bounded uniformly continuous functions on $\mathbb{R}_+$ with supremum norm.
  Define $(T(t)f)(\theta) = f(\theta + t), f \in X, \theta \geq 0, t \geq 0$ then $T(t)$ is a $C^0$ semigroup with generator $(Cf)(\theta) = f'(\theta)$ with domain $D(C) \equiv \{ f \in X : f$ differentiable and $f' \in X \}$
Examples

- If $X$ is the Banach space of bounded uniformly continuous functions on $\mathbb{R}_+$ with supremum norm.
  Define $(T(t)f)(\theta) = f(\theta + t)$, $f \in X$, $\theta \geq 0$, $t \geq 0$ then $T(t)$ is a $C^0$ semigroup with generator $(Cf)(\theta) = f'(\theta)$ with domain $D(C) \equiv \{ f \in X : f$ differentiable and $f' \in X \}$

- If $C$ is a bounded operator on a Banach space $X$ then $T(t) = e^{Ct} = \sum_{n=0}^{\infty} \frac{(Ct)^n}{n!}$ is a $C^0$ semigroup.
Examples

- If $X$ is the Banach space of bounded uniformly continuous functions on $\mathbb{R}_+$ with supremum norm. Define $(T(t)f)(\theta) = f(\theta + t), f \in X, \theta \geq 0, t \geq 0$ then $T(t)$ is a $C^0$ semigroup with generator $(Cf)(\theta) = f'(\theta)$ with domain $D(C) \equiv \{f \in X : f \text{ differentiable and } f' \in X\}$

- If $C$ is a bounded operator on a Banach space $X$ then $T(t) = e^{Ct} = \sum_{n=0}^{\infty} \frac{(Ct)^n}{n!}$ is a $C^0$ semigroup. Its generator is $C$. 

Martino Pitruzzella
Application of semigroup theory to reaction-diffusion equations
Properties of $C^0$ semigroups

If $T(t)$ is a $C^0$ semigroup on $X$ then:
Properties of $C^0$ semigroups

If $T(t)$ is a $C^0$ semigroup on $X$ then:

- $\exists \omega \in \mathbb{R}$ and $M \geq 1$ such that $\| T(t) \| \leq Me^{\omega t}, \forall t \geq 0$
Properties of $C^0$ semigroups

If $T(t)$ is a $C^0$ semigroup on $X$ then:

- $\exists \omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}, \forall t \geq 0$
- $t \mapsto T(t)x$ is continuous on $[0, \infty), \forall x \in X$
Properties of $C^0$ semigroups

If $T(t)$ is a $C^0$ semigroup on $X$ then:

- $\exists \omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$, $\forall t \geq 0$
- $t \mapsto T(t)x$ is continuous on $[0, \infty)$, $\forall x \in X$
- If $C$ is the infinitesimal generator of $T(t)$ then:
Properties of $C^0$ semigroups

If $T(t)$ is a $C^0$ semigroup on $X$ then:

- $\exists \omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}, \forall t \geq 0$
- $t \mapsto T(t)x$ is continuous on $[0, \infty), \forall x \in X$
- If $C$ is the infinitesimal generator of $T(t)$ then:
  - $T(t)x \in D(C)$ and $\frac{d}{dt}(T(t)x) = CT(t)x = T(t)Cx$
  - $\forall x \in D(C), t \in \mathbb{R}_+$
So the question is: given a (closed) operator $C$, is it the generator of a $C^0$ semigroup?
So the question is: given a (closed) operator $C$, is it the generator of a $C^0$ semigroup? answer:
Theorem (Hille-Yoshida, contraction case):
A linear operator $C$ on a Banach space $X$ is the generator of a $C^0$ semigroup of contractions on $X$ $\iff$
So the question is: given a (closed) operator $C$, is it the generator of a $C^0$ semigroup? answer:

Theorem (Hille-Yoshida, contraction case):

A linear operator $C$ on a Banach space $X$ is the generator of a $C^0$ semigroup of contractions on $X$ \iff

- $C$ is closed and densely defined
Hille-Yoshida theorem

So the question is: given a (closed) operator $C$, is it the generator of a $C^0$ semigroup? Answer:

Theorem (Hille-Yoshida, contraction case):
A linear operator $C$ on a Banach space $X$ is the generator of a $C^0$ semigroup of contractions on $X$ $\iff$

- $C$ is closed and densely defined
- $(0, \infty) \subset \rho(C)$, the resolvent set of $C$, and
  $\|R(\lambda)\| = \|(\lambda I - C)^{-1}\| \leq \lambda^{-1}, \forall \lambda > 0$
More Definitions and theorems

Definition: let $H$ be a Hilbert space. A linear operator $A$ with domain $D(A) \subset H$ is said to be dissipative
More Definitions and theorems

Definition: let $H$ be a Hilbert space. A linear operator $A$ with domain $D(A) \subset H$ is said to be dissipative if

$$<Ax,x> + <x,Ax> \leq 0, \forall x \in D(A)$$
More Definitions and theorems

Definition: let $H$ be a Hilbert space. A linear operator $A$ with domain $D(A) \subset H$ is said to be dissipative if

$$< Ax, x > + < x, Ax > \leq 0, \forall x \in D(A)$$

Theorem

- If $T(t)$ is a $C^0$ contraction semigroup on a Hilbert space $H \Rightarrow$ its infinitesimal generator is dissipative.
More Definitions and theorems

Definition: let $H$ be a Hilbert space. A linear operator $A$ with domain $D(A) \subset H$ is said to be dissipative if

$$<Ax, x> + <x, Ax> \leq 0, \forall x \in D(A)$$

Theorem

- If $T(t)$ is a $C^0$ contraction semigroup on a Hilbert space $H \Rightarrow$ its infinitesimal generator is dissipative.
- If $A$ is dissipative, densely defined and the range of $I - A$ is dense in $H \Rightarrow$ the closure $\overline{A}$ of $A$ generates a contraction semigroup.
Definition: let $H$ be a Hilbert space. A linear operator $A$ with domain $D(A) \subset H$ is said to be dissipative if
$$< Ax, x > + < x, Ax > \leq 0, \forall x \in D(A)$$

Theorem

- If $T(t)$ is a $C^0$ contraction semigroup on a Hilbert space $H \Rightarrow$ its infinitesimal generator is dissipative.
- If $A$ is dissipative, densely defined and the range of $I - A$ is dense in $H \Rightarrow$ the closure $\overline{A}$ of $A$ generates a contraction semigroup.
- If $A$ is densely defined and both $A$ and its adjoint $A^*$ are densely defined and dissipative $\Rightarrow$ the closure $\overline{A}$ of $A$ generates a contraction semigroup.
More Definitions and theorems

Definition: let $H$ be a Hilbert space. A linear operator $A$ with domain $D(A) \subset H$ is said to be dissipative if

$$< Ax, x > + < x, Ax > \leq 0, \forall x \in D(A)$$

Theorem

- If $T(t)$ is a $C^0$ contraction semigroup on a Hilbert space $H \Rightarrow$ its infinitesimal generator is dissipative.

- If $A$ is dissipative, densely defined and the range of $I - A$ is dense in $H \Rightarrow$ the closure $\overline{A}$ of $A$ generates a contraction semigroup.

- If $A$ is densely defined and both $A$ and its adjoint $A^*$ are densely defined and dissipative $\Rightarrow$ the closure $\overline{A}$ of $A$ generates a contraction semigroup.
More Definitions and theorems

We will need also the following:
We will need also the following:

**Theorem:**
Suppose \( f : \Omega \times \mathbb{R}^n \to \mathbb{R} \) is smooth, where \( \Omega \subset \mathbb{R}^m \) is bounded and \( \partial \Omega \) is smooth and \( k > \frac{m}{2} \).
More Definitions and theorems

We will need also the following:

Theorem:

Suppose $f : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth, where $\Omega \subset \mathbb{R}^m$ is bounded and $\partial \Omega$ is smooth and $k > \frac{m}{2}$ ⇒ $F : s \rightarrow f(\cdot, s(\cdot))$ from $[H^k(\Omega)]^n \rightarrow H^k(\Omega)$ is well defined and smooth. Here $H^k(\Omega)$ is the Sobolev space of (equivalence classes of) functions $u : \Omega \rightarrow \mathbb{R}$ that have weak derivatives up to and including order $k$ in $L^2(\Omega)$ with the norm

$$|u|_{k, \Omega}^2 = \left[ \int_{\Omega} \sum_{|\alpha| \leq k} |D^\alpha u|^2 dx \right]^{1/2}$$
More Definitions and theorems

Definition:
Let $0 < \theta \leq \frac{\pi}{2}$ and $\Delta_\theta = \{ \xi \in \mathbb{C} | \xi \neq 0, |\arg \xi| < \theta \}$. 
More Definitions and theorems

Definition:
Let $0 < \theta \leq \frac{\pi}{2}$ and $\Delta_\theta = \{\xi \in \mathbb{C} | \xi \neq 0, |\arg \xi| < \theta \}$. A semigroup $T(t)$ is said to be analytic of angle $\theta \in (0, \frac{\pi}{2}]$ if
Definition:
Let $0 < \theta \leq \frac{\pi}{2}$ and $\Delta_\theta = \{\xi \in \mathbb{C} \mid \xi \neq 0, |\arg\xi| < \theta\}$. A semigroup $T(t)$ is said to be analytic of angle $\theta \in (0, \frac{\pi}{2}]$ if

- $T(0) = I$ and $T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2)$ for all $\xi_{1,2} \in \Delta_\delta$
More Definitions and theorems

Definition:
Let $0 < \theta \leq \frac{\pi}{2}$ and $\Delta_\theta = \{ \xi \in \mathbb{C} | \xi \neq 0, \arg \xi < \theta \}$. A semigroup $T(t)$ is said to be analytic of angle $\theta \in (0, \frac{\pi}{2}]$ if

- $T(0) = I$ and $T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2)$ for all $\xi_{1,2} \in \Delta_\delta$
- $\xi \mapsto T(\xi)$ is analytic in the sector $\Delta_\theta$
More Definitions and theorems

Definition:
Let $0 < \theta \leq \frac{\pi}{2}$ and $\Delta_\theta = \{ \xi \in \mathbb{C} | \xi \neq 0, |\arg \xi| < \theta \}$. A semigroup $T(t)$ is said to be analytic of angle $\theta \in (0, \frac{\pi}{2}]$ if

- $T(0) = I$ and $T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2)$ for all $\xi_{1,2} \in \Delta_\delta$
- $\xi \mapsto T(\xi)$ is analytic in the sector $\Delta_\theta$
- $|T(\xi)x - x| \to 0$ as $|\xi| \to 0$ in any closed subsector of $\Delta_\theta$, $\forall x \in X$
More Definitions and theorems

Theorem:
Suppose $A$ is a closed operator with dense domain such that:

- There exists $\delta \in (0, \frac{\pi}{2}]$ such that the resolvent of $A$ contains the sector $\Delta_{\frac{\pi}{2} + \delta}$.
- For each $\epsilon \in (0, \delta)$ there exists $M_{\epsilon} > 1$ such that $\|R(\lambda, A)\| \leq M_{\epsilon} / |\lambda|$ for all $0 \neq \lambda \in \Delta_{\frac{\pi}{2} + \delta} - \epsilon$.

In this case $A$ is called a sectorial operator of angle $\delta$.

$\Rightarrow$ $A$ generates a bounded analytic semigroup of angle $\delta$. 
Theorem:
Suppose $A$ is a closed operator with dense domain such that:

- There exists $\delta \in (0, \pi/2]$ such that the resolvent of $A$ contains the sector $\Delta \frac{\pi}{2} + \delta$
Theorem:
Suppose $A$ is a closed operator with dense domain such that:

- There exists $\delta \in (0, \frac{\pi}{2}]$ such that the resolvent of $A$ contains the sector $\Delta \frac{\pi}{2} + \delta$
- For each $\epsilon \in (0, \delta)$ there exists $M_\epsilon > 1$ such that $\| R(\lambda, A) \| \leq M_\epsilon / |\lambda|$ for all $0 \neq \lambda \in \overline{\Delta} \frac{\pi}{2} + \delta - \epsilon$

In this case $A$ is called a sectorial operator of angle $\delta$. 
More Definitions and theorems

Theorem:
Suppose $A$ is a closed operator with dense domain such that:

- There exists $\delta \in (0, \frac{\pi}{2}]$ such that the resolvent of $A$ contains the sector $\Delta \frac{\pi}{2} + \delta$
- For each $\epsilon \in (0, \delta)$ there exists $M_\epsilon > 1$ such that $\| R(\lambda, A) \| \leq M_\epsilon / |\lambda|$ for all $0 \neq \lambda \in \overline{\Delta \frac{\pi}{2} + \delta - \epsilon}$

In this case $A$ is called a sectorial operator of angle $\delta$.
$\Rightarrow$ $A$ generates a bounded analytic semigroup of angle $\delta$. 
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$

Theorem:
A $C^0$ semigroup with generator $C$ is compact $\Leftrightarrow$
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$

Theorem:
A $C^0$ semigroup with generator $C$ is compact $\iff$

$\implies t \to T(t)$ is norm continuous on $(0, \infty)$
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$

Theorem:
A $C^0$ semigroup with generator $C$ is compact $\iff$
- $t \to T(t)$ is norm continuous on $(0, \infty)$
- $R(\lambda, C) = (\lambda I - C)^{-1}$ is compact for some $\lambda \in \rho(C)$ (i.e. $\forall \lambda \in \rho(C)$)
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$

Theorem:
A $C^0$ semigroup with generator $C$ is compact $\iff$
- $t \to T(t)$ is norm continuous on $(0, \infty)$
- $R(\lambda, C) = (\lambda I - C)^{-1}$ is compact for some $\lambda \in \rho(C)$ (i.e. $\forall \lambda \in \rho(C)$)

Theorem:
Suppose $C$ is the generator of a $C^0$ semigroup $T(t)$ and $A \in L(Z)$ is a bounded operator $\Rightarrow$
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$

Theorem:
A $C^0$ semigroup with generator $C$ is compact $\iff$

- $t \to T(t)$ is norm continuous on $(0, \infty)$
- $R(\lambda, C) = (\lambda I - C)^{-1}$ is compact for some $\lambda \in \rho(C)$ (i.e. $\forall \lambda \in \rho(C)$)

Theorem:
Suppose $C$ is the generator of a $C^0$ semigroup $T(t)$ and $A \in L(Z)$ is a bounded operator $\Rightarrow C + A$ generates a $C^0$ semigroup $S$
More Definitions and theorems

Definition:
A $C^0$ semigroup of bounded linear operators $T(t)$ is said to be compact if $T(t)$ is compact $\forall t > 0$

Theorem:
A $C^0$ semigroup with generator $C$ is compact $\iff$

$\quad t \rightarrow T(t)$ is norm continuous on $(0, \infty)$

$\quad R(\lambda, C) = (\lambda I - C)^{-1}$ is compact for some $\lambda \in \rho(C)$ (i.e. $\forall \lambda \in \rho(C)$)

Theorem:
Suppose $C$ is the generator of a $C^0$ semigroup $T(t)$ and $A \in L(Z)$ is a bounded operator $\Rightarrow C + A$ generates a $C^0$ semigroup $S$

$\quad$ if $T$ is analytic $\Rightarrow S$ is analytic

$\quad$ if $T$ is compact $\Rightarrow S$ is compact
First we consider the Laplace operator:
\[ \Delta u = \left( \frac{\partial^2}{\partial x^2} + \cdots + \frac{\partial^2}{\partial x^m} \right) u. \]
where \( u \) is a function on \( \Omega \), with \( u = 0 \) on \( \partial \Omega \).

The Laplace operator can be extended to a closed, self-adjoint operator \( A : D(A) \subset L^2(\Omega) \to L^2(\Omega) \) with dense domain \( D(A) \) given by the closure of the set:
\[ C^2_0(\Omega) = \{ u \in C^2(\Omega) | u = 0 \text{ on } \partial \Omega \}. \]

The space \( L^2(\Omega) \) is a Hilbert space and \( A \) is dissipative because for \( u \in C^2_0(\Omega) \) we have:
\[ \langle \Delta u, u \rangle \leq 0 \Rightarrow \langle Au, u \rangle \leq 0 \text{ for } u \in D(A). \]

Therefore \( A \) generates a contraction semigroup on \( L^2(\Omega) \).

Moreover the semigroup generated by \( A \) is also analytic and compact.
First we consider the Laplace operator:

\[ \Delta u = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right) u. \]

where \( u \) is a function on \( \Omega \), with \( u = 0 \) on \( \partial \Omega \).
First we consider the Laplace operator:
\[ \Delta u = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right) u. \]
where \( u \) is a function on \( \Omega \), with \( u = 0 \) on \( \partial \Omega \). The Laplace operator can be extended to a closed, self-adjoint operator \( A : D_A \subset L^2(\Omega) \rightarrow L^2(\Omega) \) with dense domain \( D_A \) given by the closure of the set:
\[ C_0^2(\Omega) = \left\{ u \in C^2(\Omega) | u = 0 \text{ on } \partial \Omega \right\} \text{ in } H^2(\Omega) \]
First we consider the Laplace operator: 
\[ \Delta u = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right) u. \]
where \( u \) is a function on \( \Omega \), with \( u = 0 \) on \( \partial \Omega \). The Laplace operator can be extended to a closed, self-adjoint operator 
\( A : D_A \subset L^2(\Omega) \to L^2(\Omega) \) with dense domain \( D_A \) given by the closure of the set:
\[ C^2_0(\Omega) = \left\{ u \in C^2(\Omega) | u = 0 \text{ on } \partial \Omega \right\} \text{ in } H^2(\Omega) \]
The space \( L^2(\Omega) \) is a Hilbert space and \( A \) is dissipative because for \( u \in C^2_0(\Omega) \) we have: \( < \Delta u, u > \leq 0 \Rightarrow < Au, u > \leq 0 \) for \( u \in D_A \).
First we consider the Laplace operator:
\[ \Delta u = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right) u. \]
where \( u \) is a function on \( \Omega \), with \( u = 0 \) on \( \partial \Omega \). The Laplace operator can be extended to a closed, self-adjoint operator \( A : D_A \subset L^2(\Omega) \to L^2(\Omega) \) with dense domain \( D_A \) given by the closure of the set:
\[ C^2_0(\overline{\Omega}) = \left\{ u \in C^2(\overline{\Omega}) \mid u = 0 \text{ on } \partial \Omega \right\} \text{ in } H^2(\Omega) \]
The space \( L^2(\Omega) \) is a Hilbert space and \( A \) is dissipative because for \( u \in C^2_0(\overline{\Omega}) \) we have: \( \langle \Delta u, u \rangle \leq 0 \Rightarrow \langle Au, u \rangle \leq 0 \) for \( u \in D_A \). Therefore \( A \) generates a contraction semigroup on \( L^2(\Omega) \).
First we consider the Laplace operator:
\[ \Delta u = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right) u. \]
where \( u \) is a function on \( \Omega \), with \( u = 0 \) on \( \partial \Omega \). The Laplace operator can be extended to a closed, self-adjoint operator \( A : D_A \subset L^2(\Omega) \to L^2(\Omega) \) with dense domain \( D_A \) given by the closure of the set:
\[ C^2_0(\Omega) = \left\{ u \in C^2(\Omega) \mid u = 0 \text{ on } \partial \Omega \right\} \text{ in } H^2(\Omega) \]
The space \( L^2(\Omega) \) is a Hilbert space and \( A \) is dissipative because for \( u \in C^2_0(\Omega) \) we have: \( < \Delta u, u > \leq 0 \Rightarrow < Au, u > \leq 0 \) for \( u \in D_A \). Therefore \( A \) generates a contraction semigroup on \( L^2(\Omega) \).
Moreover the semigroup generated by \( A \) is also analytic and compact.
The domain $D_A$ is a Banach space with the same norm of $H^2(\Omega)$. We define now $\tilde{A}$ as the restriction of $A$ to the subspace $D_A^2 = \{ u \in D_A | Au \in D_A \}$ and $\tilde{T}(t)$ the restriction of $T(t)$ to the subspace $D_A$ of $L^2(\Omega)$. 
The domain $D_A$ is a Banach space with the same norm of $H^2(\Omega)$. We define now $\tilde{A}$ as the restriction of $A$ to the subspace $D_{A^2} = \{u \in D_A | Au \in D_A\}$ and $\tilde{T}(t)$ the restriction of $T(t)$ to the subspace $D_A$ of $L^2(\Omega)$. We do this because the substitution operator associated with the non-linearity is smooth, hence the non-linear part is well defined and smooth in $D_A$. 
Laplace operator

The domain $D_A$ is a Banach space with the same norm of $H^2(\Omega)$. We define now $\tilde{A}$ as the restriction of $A$ to the subspace $D_{A^2} = \{u \in D_A | Au \in D_A\}$ and $\tilde{T}(t)$ the restriction of $T(t)$ to the subspace $D_A$ of $L^2(\Omega)$. We do this because the substitution operator associated with the non-linearity is smooth, hence the non-linear part is well defined and smooth in $D_A$.

In the case of Neumann boundary conditions the result is valid as well and $A$ defined as before still generates a contraction semigroup.
Abstract evolution equations

Next, given a Banach space $X$, we consider evolution equations of the form:

$$\dot{u} = Cu + f(u),$$

$u(0) = u_0, u, u_0 \in X$ where $C$ is the generator of a $C_0$ semigroup $T(t)$ on $X$ and $f: X \to X$ is smooth of class $C^k$.

The solution to this equation satisfies the integral equation (Duhamel's formula):

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s)) \, ds$$

Since $f$ is locally Lipschitz and $\|T(t)\| \leq Me^{\omega t}$, Picard iteration shows that $C + f$ generates a non-linear $C_0$ semigroup $F(t)$.

Since the integral equation above is not defined for $\forall t \in \mathbb{R}^+$, this semigroup is only defined on an interval $[0, \alpha)$. 

Application of semigroup theory to reaction-diffusion equations
Abstract evolution equations

Next, given a Banach space $X$, we consider evolution equations of the form: $\dot{u} = Cu + f(u)$, $u(0) = u_0$, $u, u_0 \in X$
Abstract evolution equations

Next, given a Banach space $X$, we consider evolution equations of the form: $\dot{u} = Cu + f(u)$, $u(0) = u_0$, $u, u_0 \in X$ where $C$ is the generator of a $C^0$ semigroup $T(t)$ on $X$ and $f : X \to X$ is smooth of class $C^k$. 
Abstract evolution equations

Next, given a Banach space $X$, we consider evolution equations of the form: $\dot{u} = Cu + f(u)$, $u(0) = u_0$, $u, u_0 \in X$ where $C$ is the generator of a $C^0$ semigroup $T(t)$ on $X$ and $f : X \to X$ is smooth of class $C^k$.

The solution to this equation satisfies the integral equation (Duhamel’s formula):

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(u(s))ds$$
Next, given a Banach space $X$, we consider evolution equations of the form: $\dot{u} = Cu + f(u)$, $u(0) = u_0$, $u, u_0 \in X$ where $C$ is the generator of a $C^0$ semigroup $T(t)$ on $X$ and $f : X \rightarrow X$ is smooth of class $C^k$. The solution to this equation satisfies the integral equation (Duhamel’s formula):

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(u(s))ds$$

Since $f$ is locally Lipschitz and $\|T(t)\| \leq Me^{\omega t}$, Picard iteration shows that $C + f$ generates a non-linear $C^0$ semigroup $F(t)$. 
Next, given a Banach space $X$, we consider evolution equations of the form: \[ \dot{u} = Cu + f(u), \quad u(0) = u_0, \quad u, u_0 \in X \] where $C$ is the generator of a $C^0$ semigroup $T(t)$ on $X$ and $f : X \to X$ is smooth of class $C^k$.

The solution to this equation satisfies the integral equation (Duhamel’s formula):
\[ u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s))\,ds \]

Since $f$ is locally Lipschitz and $\|T(t)\| \leq Me^{\omega t}$, Picard iteration shows that $C + f$ generates a non-linear $C^0$ semigroup $F(t)$. Since the integral equation above is not defined $\forall t \in \mathbb{R}_+$, this semigroup is only defined on an interval $[0, \alpha)$. 

Application of semigroup theory to reaction-diffusion equations

Martino Pitruzzella

Abstract evolution equation

Laplace operator

Principle of linearized stability

Example: Turing instability on interval

Reaction-diffusion equations
We consider now an $n$-component reaction-diffusion system:

$$
\frac{d}{dt}u_i = d_i \Delta u_i + \sum_{j=1}^{n} c_{ij} u_j + f_i(u), \quad (i = 1, \ldots, n)
$$

on a bounded domain $\Omega \subset \mathbb{R}^m$ ($m \leq 3$) with Dirichlet or Neumann conditions on its smooth boundary $\partial \Omega$, where $d_i$ and $c_{ij}$ are real numbers, $d_i > 0$, $f_i: \mathbb{R}^n \to \mathbb{R}$ are smooth functions in $u_1, \ldots, u_n$ with $f_i(0) = 0$.
We consider now an $n$-component reaction-diffusion system:

$$\frac{d}{dt} u_i = d_i \Delta u_i + \sum_{j=1}^{n} c_{ij} u_j + f_i(u), \quad (i = 1, \cdots, n).$$

on a bounded domain $\Omega \subset \mathbb{R}^m (m \leq 3)$. 

---

**Reaction-diffusion equations**

- Aim of the talk, introduction and motivation
- Summary of semigroup theory
- How the theory is applied
- Principle of linearized stability
- Example: Turing instability on interval
We consider now an $n$-component reaction-diffusion system:

$$\frac{d}{dt} u_i = d_i \Delta u_i + \sum_{j=1}^{n} c_{ij} u_j + f_i(u), \quad (i = 1, \cdots, n).$$

on a bounded domain $\Omega \subset \mathbb{R}^m (m \leq 3)$ with Dirichlet or Neumann conditions on its smooth boundary $\partial \Omega$. 

We consider now an $n$-component reaction-diffusion system:

$$\frac{d}{dt} u_i = d_i \Delta u_i + \sum_{j=1}^{n} c_{ij} u_j + f_i(u), \quad (i = 1, \cdots, n).$$

on a bounded domain $\Omega \subset \mathbb{R}^m \,(m \leq 3)$ with Dirichlet or Neumann conditions on its smooth boundary $\partial \Omega$ where $d_i$ and $c_{ij}$ are real numbers, $d_i > 0$.
We consider now an $n$-component reaction-diffusion system:

$$\frac{d}{dt} u_i = d_i \Delta u_i + \sum_{j=1}^{n} c_{ij} u_j + f_i(u), \quad (i = 1, \ldots, n).$$

on a bounded domain $\Omega \subset \mathbb{R}^m (m \leq 3)$ with Dirichlet or Neumann conditions on its smooth boundary $\partial \Omega$ where $d_i$ and $c_{ij}$ are real numbers, $d_i > 0$, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are smooth functions in $u_1, \ldots, u_n$ with $f_i(0) = 0$. 
We write this system as \( \dot{u} = D\Delta u + Cu + f(u) \) where 
\( u = (u_1, ..., u_n)^T \), 
\( D = \text{diag}(d_1, ..., d_n) \) is a diagonal matrix, 
\( C = (c_{ij}) \) and 
\( f = (f_1, ..., f_n)^T \).
Reaction-diffusion equations

We write this system as $\dot{u} = D\Delta u + Cu + f(u)$ where $u = (u_1, .., u_n)^T$, $D = \text{diag}(d_1, .., d_n)$ is a diagonal matrix, $C = (c_{ij})$ and $f = (f_1, .., f_n)^T$.

Let $\tilde{A}_i : D_{A_i}^2 \rightarrow D_{A_i}$ be the operator defined as before, i.e. the extension of $d_i\Delta$ restricted to $D_{A_i}^2$. 
We write this system as $\dot{u} = D\Delta u + Cu + f(u)$ where
\[ u = (u_1, \ldots, u_n)^T, \quad D = \text{diag}(d_1, \ldots, d_n) \text{ is a diagonal matrix,} \]
\[ C = (c_{ij}) \text{ and } f = (f_1, \ldots, f_n)^T. \]

Let $\tilde{A}_i : D_{A_i} \rightarrow D_{A_i}$ be the operator defined as before, i.e. the extension of $d_i\Delta$ restricted to $D_{A_i}$. Set also the new space $X = D_{A_1} \times \cdots \times D_{A_n}$ and $\tilde{A} = \tilde{A}_1 \times \cdots \times \tilde{A}_n$. So we have that $\tilde{A}$ generates a compact analytic semigroup on $X$. 

---

**Reaction-diffusion equations**

Martino Pitruzzella

Application of semigroup theory to reaction-diffusion equations
Reaction-diffusion equations

We write this system as \( \dot{u} = D\Delta u + Cu + f(u) \) where 
\( u = (u_1, \ldots, u_n)^T \), \( D = \text{diag}(d_1, \ldots, d_n) \) is a diagonal matrix, 
\( C = (c_{ij}) \) and \( f = (f_1, \ldots, f_n)^T \).

Let \( \tilde{A}_i : D_{A_i}^2 \to D_{A_i} \) be the operator defined as before, i.e. the 
extension of \( d_i\Delta \) restricted to \( D_{A_i}^2 \). Set also the new space 
\( X = D_{A_1} \times \cdots \times D_{A_n} \) and \( \tilde{A} = \tilde{A}_1 \times \cdots \times \tilde{A}_n \). So we have that \( \tilde{A} \) generates a compact analytic semigroup on \( X \). Then also \( \tilde{A} + C \) generates a compact analytic semigroup.
We write this system as \( \dot{u} = D\Delta u + Cu + f(u) \) where 
\( u = (u_1, .., u_n)^T \), \( D = \text{diag}(d_1, .., d_n) \) is a diagonal matrix, 
\( C = (c_{ij}) \) and \( f = (f_1, .., f_n)^T \).

Let \( \tilde{A}_i : D_{A_i^2} \to D_{A_i} \) be the operator defined as before, i.e. the extension of \( d_i\Delta \) restricted to \( D_{A_i^2} \). Set also the new space 
\( X = D_{A_1} \times \cdots \times D_{A_n} \) and \( \tilde{A} = \tilde{A}_1 \times \cdots \times \tilde{A}_n \). So we have that \( \tilde{A} \) generates a compact analytic semigroup on \( X \). Then also \( \tilde{A} + C \) generates a compact analytic semigroup. By the theorem above we have that \( f : X \to X \) is a smooth function. So by Picard Iteration we have that \( D\Delta + C + f \) generates a \( C^0 \) semigroup.
Principle of linearized stability

The principle of linearized stability in the finite dimensional case says that, if 0 is an equilibrium of the system of differential equations $\dot{u} = f(u)$ and all the eigenvalues of the Jacobian matrix $Df$ have real part less than zero, then the zero solution is stable. We see now how this result is also valid for evolution equations under some assumptions.
Consider the equation: \( \dot{u}(t) = A(u(t)) + f(u(t)), u(0) = u_0, t > 0 \)
where \( A \) is a sectorial operator on \( X \) and \( f : X \to X \) is smooth and suppose 0 is a solution. We have \( u(t) = F(t)u_0 \), where \( F(t) \) is the non linear semigroup associated with the equation above.
Consider the equation: $\dot{u}(t) = A(u(t)) + f(u(t)) , u(0) = u_0, \ t > 0$
where $A$ is a sectorial operator on $X$ and $f : X \to X$ is smooth and
suppose 0 is a solution. We have $u(t) = F(t)u_0$, where $F(t)$ is the non linear semigroup associated with the equation above.
Definition: The zero solution of the equation above is called stable in $X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that: $u_0 \in X, \|u(0)\| \leq \delta \Rightarrow$
the solution is defined $\forall t > 0 \ , \ \|u(t)\| \leq \epsilon , \ \forall t \geq 0$. 

Consider the equation: \( \dot{u}(t) = A(u(t)) + f(u(t)), u(0) = u_0, t > 0 \)
where \( A \) is a sectorial operator on \( X \) and \( f : X \to X \) is smooth and suppose 0 is a solution. We have \( u(t) = F(t)u_0 \), where \( F(t) \) is the non linear semigroup associated with the equation above.

Definition: The zero solution of the equation above is called stable in \( X \) if \( \forall \epsilon > 0, \exists \delta > 0 \) such that: \( u_0 \in X, \|u(0)\| \leq \delta \Rightarrow \)
the solution is defined \( \forall t > 0, \|u(t)\| \leq \epsilon, \forall t \geq 0. \)

The zero solution is called asymptotically stable is it is stable and moreover \( \exists \delta_0 > 0 \) such that if \( \|u(0)\| \leq \delta_0 \) then
\( \lim_{t \to \infty} \|u(t)\| = 0 \)
Principle of linearized stability

The spectral bound of a sectorial operator $A$ is defined as:
\[ s(A) = \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}. \]
Principle of linearized stability

The spectral bound of a sectorial operator $A$ is defined as:

$$s(A) = \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}.$$ 

Theorem (Principle of linear stability):
Suppose $s(A) < 0$ and $F : X \to X$ is smooth in a neighborhood of 0.

Therefore the zero solution is asymptotically stable.
Principle of linearized stability

The spectral bound of a sectorial operator $A$ is defined as:
$s(A) = \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}$.

Theorem (Principle of linear stability):
Suppose $s(A) < 0$ and $F : X \to X$ is smooth in a neighborhood of 0.
Then $\forall \omega \in [0, -s(A)]$ there exists positive constants $M = M(\omega), r = r(\omega)$ such that if $u_0 \in X, u_0 \geq r \Rightarrow$ we have that the solution is defined $\forall t > 0$ and $\|u(t)\| \leq Me^{-\omega t} \|u_0\|, t \geq 0$
Therefore the zero solution is asymptotically stable.
Turing instability

Let's consider the following reaction-diffusion system of two coupled equations on the interval $[0, \pi]$ for $u = u(t, x)$:

$$\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + f_1(u, v) \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + f_2(u, v)
\end{align*}$$

with $u(t, 0) = u(t, \pi) = 0$ and $f_1(0, 0) = f_2(0, 0) = 0$.

Then $(u, v) = (0, 0)$ is an homogeneous solution, that is a solution of:

$$\begin{align*}
\frac{\partial u}{\partial t} &= f_1(u, v) \\
\frac{\partial v}{\partial t} &= f_2(u, v)
\end{align*}$$

The Jacobian matrix is:

$$Df = \begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}$$
Turing instability

Let's consider the following reaction-diffusion system of two coupled equations on the interval $[0, \pi]$ for $u = u(t, x)$:

$$
\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + f_1(u, v)
$$

$$
\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + f_2(u, v)
$$

with $u(t, 0) = u(t, \pi) = 0$ and $f_1$ and $f_2$ are smooth functions. This is a particular case covered by the previous theory so the system defines a nonlinear local semigroup on $H^2([0, \pi])$. Assume that $f_1(0, 0) = 0 = f_2(0, 0)$.
Turing instability

Let’s consider the following reaction-diffusion system of two coupled equations on the interval \([0, \pi]\) for \(u = u(t, x)\):
\[
\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + f_1(u, v) \\
\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + f_2(u, v)
\]
with \(u(t, 0) = u(t, \pi) = 0\) and \(f_1\) and \(f_2\) are smooth functions. This is a particular case covered by the previous theory so the system defines a nonlinear local semigroup on \(H^2([0, \pi])\). Assume that \(f_1(0, 0) = 0 = f_2(0, 0)\) then \((u, v) = (0, 0)\) is an homogeneous solution, that is a solution of:
\[
\frac{\partial u}{\partial t} = f_1(u, v) \\
\frac{\partial v}{\partial t} = f_2(u, v)
\]
Let’s consider the following reaction-diffusion system of two coupled equations on the interval $[0, \pi]$ for $u = u(t, x)$:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + f_1(u, v) \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + f_2(u, v)
\end{align*}
\]

with $u(t, 0) = u(t, \pi) = 0$ and $f_1$ and $f_2$ are smooth functions. This is a particular case covered by the previous theory so the system defines a nonlinear local semigroup on $H^2([0, \pi])$.

Assume that $f_1(0, 0) = 0 = f_2(0, 0)$ then $(u, v) = (0, 0)$ is an homogeneous solution, that is a solution of:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= f_1(u, v) \\
\frac{\partial v}{\partial t} &= f_2(u, v)
\end{align*}
\]

The Jacobian matrix is: $Df = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$
Turing instability

The eigenvalues are found by:
\[ \lambda^2 - (\text{Tr } M)\lambda + \text{Det } M = 0 = \lambda^2 - (m_{11} + m_{22})\lambda + m_{11}m_{22} - m_{21}m_{12} \]
Turing instability

The eigenvalues are found by:
\[ \lambda^2 - (\text{Tr } M)\lambda + \text{Det } M = 0 = \lambda^2 - (m_{11} + m_{22})\lambda + m_{11} m_{22} - m_{21} m_{12} \]
Suppose the equilibrium is stable, that is: \( m_{11} + m_{22} < 0 \) and \( m_{11} m_{22} - m_{21} m_{12} > 0 \)
Turing instability

The eigenvalues are found by:
\[
\lambda^2 - (\text{Tr } M)\lambda + \text{Det } M = 0 = \lambda^2 - (m_{11} + m_{22})\lambda + m_{11}m_{22} - m_{21}m_{12}
\]
Suppose the equilibrium is stable, that is: \(m_{11} + m_{22} < 0\) and \(m_{11}m_{22} - m_{21}m_{12} > 0\)
Consider now the linearized system with diffusion terms.
Turing instability

The eigenvalues are found by:
\[ \lambda^2 - (\text{Tr } M)\lambda + \text{Det } M = 0 = \lambda^2 - (m_{11} + m_{22})\lambda + m_{11}m_{22} - m_{21}m_{12} \]
Suppose the equilibrium is stable, that is: \( m_{11} + m_{22} < 0 \) and \( m_{11}m_{22} - m_{21}m_{12} > 0 \)
Consider now the linearized system with diffusion terms.
\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + m_{11}u + m_{12}v \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + m_{21}u + m_{22}v
\end{align*}
\]
the right hand side of this equation is \((\tilde{A} + C)(u, v)\).
Turing instability

The eigenvalues are found by:
\[
\lambda^2 - (\text{Tr } M) \lambda + \text{Det } M = 0 = \lambda^2 - (m_{11} + m_{22}) \lambda + m_{11} m_{22} - m_{21} m_{12}
\]
Suppose the equilibrium is stable, that is: \(m_{11} + m_{22} < 0\) and \(m_{11} m_{22} - m_{21} m_{12} > 0\)
Consider now the linearized system with diffusion terms.
\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + m_{11} u + m_{12} v \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + m_{21} u + m_{22} v
\end{align*}
\]
the right hand side of this equation is \((\tilde{A} + C)(u, v)\).
The spectrum of this operator consists of eigenvalues satisfying for all integer \(k\) the equation:
\[
\begin{vmatrix}
m_{11} - \lambda - k^2 d_1 & m_{12} \\
m_{21} & m_{22} - \lambda - k^2 d_2
\end{vmatrix} = 0
\]
Turing instability

So we get: $\lambda^2 + \lambda \left[ k^2(d_1 + d_2 - (m_{11} + m_{22})) \right] + h(k^2) = 0$

where $h(k^2) = k^4 d_1 d_2 - k^2 (m_{11} d_2 + m_{22} d_1) + (m_{11} m_{22} - m_{21} m_{12})$. 
Turing instability

So we get: \[ \lambda^2 + \lambda \left[ k^2 (d_1 + d_2 - (m_{11} + m_{22})) \right] + h(k^2) = 0 \]
where \[ h(k^2) = k^4 d_1 d_2 - k^2 (m_{11} d_2 + m_{22} d_1) + (m_{11} m_{22} - m_{21} m_{12}) \].
Since \( \text{Tr} \ M < 0 \), conditions for instability are given by the function \( h(k^2) \). That is, if \( h(k^2) < 0 \) for some \( k \), then there is instability.
Turing instability

So we get: $\lambda^2 + \lambda \left[ k^2 (d_1 + d_2 - (m_{11} + m_{22})) \right] + h(k^2) = 0$

where $h(k^2) = k^4 d_1 d_2 - k^2 (m_{11} d_2 + m_{22} d_1) + (m_{11} m_{22} - m_{21} m_{12})$.

Since $\text{Tr} \ M < 0$, conditions for instability are given by the function $h(k^2)$, That is, if $h(k^2) < 0$ for some $k$, then there is instability.

So we have: $h(k^2) = (d_1 d_2) k^4 - (m_{11} d_2 + m_{22} d_1) k^2 + \det M$

and to have $h(k^2) < 0$ the following must be satisfied:
Turing instability

So we get: $\lambda^2 + \lambda \left[ k^2(d_1 + d_2 - (m_{11} + m_{22})) \right] + h(k^2) = 0$

where $h(k^2) = k^4d_1d_2 - k^2(m_{11}d_2 + m_{22}d_1) + (m_{11}m_{22} - m_{21}m_{12})$.

Since $\text{Tr } M < 0$, conditions for instability are given by the function $h(k^2)$, That is, if $h(k^2) < 0$ for some $k$, then there is instability.

So we have: $h(k^2) = (d_1d_2)k^4 - (m_{11}d_2 + m_{22}d_1)k^2 + \text{Det } M$

and to have $h(k^2) < 0$ the following must be satisfied:

- $m_{11}d_2 + m_{22}d_1 > 0$

and the minimum of $h(k^2)$ must be below 0, this gives:

- $\frac{(m_{11}d_2 + m_{22}d_1)^2}{4d_1d_2} > \text{Det } M$
Finally we have that, to have diffusion-driven instability the following conditions must be satisfied:

- $\text{Tr } M = m_{11} + m_{22} < 0$
- $\text{Det } M = m_{11} m_{22} - m_{21} m_{12} > 0$
- $m_{11} d_2 + m_{22} d_1 > 0$
- $\frac{(m_{11} d_2 + m_{22} d_1)^2}{4 d_1 d_2} > \text{Det } M$
Turing instability

Finally we have that, to have diffusion-driven instability the following conditions must be satisfied:

- \( \text{Tr } M = m_{11} + m_{22} < 0 \)
- \( \text{Det } M = m_{11} m_{22} - m_{21} m_{12} > 0 \)
- \( m_{11} d_2 + m_{22} d_1 > 0 \)
- \( \frac{(m_{11} d_2 + m_{22} d_1)^2}{4d_1 d_2} > \text{Det } M \)

In case these conditions are satisfied we have that the spatially homogeneous stable state becomes unstable if there is integer \( k \) in a range \( k_1 < k < k_2 \) where \( k_1 \) and \( k_2 \) are given by:

\[
k_{1,2}^2 = \frac{(m_{11} d_2 + m_{22} d_1)}{2d_1 d_2} \pm \frac{\sqrt{(m_{11} d_2 + m_{22} d_1)^2 - 4d_1 d_2 \text{Det } M}}{2d_1 d_2}
\]