Pattern formation in gradient systems

Seminar on Spatio-Temporal Patterns

Stefanie Postma

Universiteit Leiden

March 7, 2012

Outline

1. Gradient Systems
   - Some properties
   - Example
   - Passage
2. The Cahn-Hilliard equation
   - Applications
   - Connection between CH-equation and gradient systems
3. The Extended Fisher-Kolmogorov and Swift-Hohenberg equation
   - The Fisher-Kolmogorov equation
   - The EFK- and SH-equation
   - General equation
   - Comparison
4. Last remarks

Gradient Systems

Definition (Gradient Systems on $\mathbb{R}^n$)

A system of differential equations of the form

$$\boxed{X' = -\text{grad } V(X),}$$

where $X = (x_1, \ldots, x_n)$ and $V : \mathbb{R}^n \to \mathbb{R}$ is a C^∞-function, and

$$\text{grad } V = \nabla V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right).$$

The vector field grad V is called the *gradient* of V.

Gradient Systems

Definition (Gradient Systems on $\mathbb{R}^n$)

A system of differential equations of the form

$$\boxed{X' = -\text{grad } V(X),}$$

where $X = (x_1, \ldots, x_n)$ and $V : \mathbb{R}^n \to \mathbb{R}$ is a C^∞-function, and

$$\text{grad } V = \nabla V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right).$$

The vector field grad V is called the *gradient* of V.

Note: the negative sign in this system is traditional. And

$$-\text{grad } V(X) = \text{ grad } (-V(X)).$$

Important equality

The following equality is fundamental:

$$DV_X(Y) = \text{ grad } V(X) \cdot Y.$$

This says that the derivative of V at X evaluated at $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is given by the dot product of the vectors grad $V(X)$ and Y.

Important equality

The following equality is fundamental:

$$DV_X(Y) = \text{ grad } V(X) \cdot Y.$$

This says that the derivative of V at X evaluated at $Y = (y_1, \dots, y_n) \in \mathbb{R}^n$ is given by the dot product of the vectors grad $V(X)$ and Y.

This follows immediately from the formula

$$DV_X(Y) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(X) y_j.$$

Important equality

The following equality is fundamental:

$$DV_X(Y) = \text{ grad } V(X) \cdot Y.$$

This says that the derivative of V at X evaluated at $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is given by the dot product of the vectors grad $V(X)$ and Y.

This follows immediately from the formula

$$DV_X(Y) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(X) y_j.$$

Let $X(t)$ be a solution of the gradient system $X' = -\text{grad } V(X)$ with $X(0) = X_0$, and let $\dot{V} : \mathbb{R}^n \to \mathbb{R}$ be the derivative of V along this solution. That is

$$\dot{V}(X) = \frac{d}{dt} V(X(t)).$$

Proposition

The function V is a Lyapunov function for the system $X' = -\text{grad}\, V(X)$. Moreover, $V(X) = 0$ if and only if X is an equilibrium point.

Proposition

The function V is a Lyapunov function for the system $X' = -\text{grad}\, V(X)$. Moreover, $V(X) = 0$ if and only if X is an equilibrium point.

Proof.

By the chain rule, we have

$$\begin{aligned} \dot{V}(X) &= DV_X(X') \\ &= \text{grad}\, V(X) \cdot (-\text{grad}\, V(X)) \\ &= -|\text{grad}\, V(X)|^2 \leq 0. \end{aligned}$$

In particular, $\dot{V}(X) = 0$ if and only if $\text{grad}\, V(X) = 0$. □

Proposition

The function V is a Lyapunov function for the system $X' = -\text{grad}\, V(X)$. Moreover, $V(X) = 0$ if and only if X is an equilibrium point.

Proof.

By the chain rule, we have

$$\begin{aligned} \dot{V}(X) &= DV_X(X') \\ &= \text{grad}\, V(X) \cdot (-\text{grad}\, V(X)) \\ &= -|\text{grad}\, V(X)|^2 \leq 0. \end{aligned}$$

In particular, $\dot{V}(X) = 0$ if and only if $\text{grad}\, V(X) = 0$. □

Remark: Lyapunov functions are scalar functions that may be used to prove the stability of an equilibrium of an ODE.

Immediate consequence:
If X^* is an isolated minimum of V, then X^* is an asymptotically stable equilibrium of the gradient system.

Immediate consequence:
If X^* is an isolated minimum of V, then X^* is an asymptotically stable equilibrium of the gradient system.

The fact that X^* is isolated guarentees that $\dot{V} < 0$ in a neighbourhood of X^* (not including X^*).

Level surfaces

To understand a gradient flow geometrically, we look at the *level surfaces* of the function $V : \mathbb{R}^n \to \mathbb{R}$. These are the subsets $V^{-1}(c)$ with $c \in \mathbb{R}$.

Level surfaces

To understand a gradient flow geometrically, we look at the *level surfaces* of the function $V : \mathbb{R}^n \to \mathbb{R}$. These are the subsets $V^{-1}(c)$ with $c \in \mathbb{R}$.

If $X \in V^{-1}(c)$ is a *regular point*, that is grad $V(X) \neq 0$, then $V^{-1}(c)$ looks like a 'surface' of dimension $n-1$ near X.

Level surfaces

To understand a gradient flow geometrically, we look at the *level surfaces* of the function $V : \mathbb{R}^n \to \mathbb{R}$. These are the subsets $V^{-1}(c)$ with $c \in \mathbb{R}$.

If $X \in V^{-1}(c)$ is a *regular point*, that is $\text{grad}\, V(X) \neq 0$, then $V^{-1}(c)$ looks like a 'surface' of dimension $n - 1$ near X.

If all points in $V^{-1}(c)$ are regular points, then we say that c is a *regular value* for V.

Level surfaces

To understand a gradient flow geometrically, we look at the *level surfaces* of the function $V : \mathbb{R}^n \to \mathbb{R}$. These are the subsets $V^{-1}(c)$ with $c \in \mathbb{R}$.

If $X \in V^{-1}(c)$ is a *regular point*, that is $\operatorname{grad} V(X) \neq 0$, then $V^{-1}(c)$ looks like a 'surface' of dimension $n-1$ near X.

If all points in $V^{-1}(c)$ are regular points, then we say that c is a *regular value* for V.

If X is a nonregular point for V, then $\operatorname{grad} V(X) = 0$, so X is a *critical point* for the function V, since all partial derivatives of V vanish at X.

Level surfaces

To understand a gradient flow geometrically, we look at the *level surfaces* of the function $V : \mathbb{R}^n \to \mathbb{R}$. These are the subsets $V^{-1}(c)$ with $c \in \mathbb{R}$.

If $X \in V^{-1}(c)$ is a *regular point*, that is $\operatorname{grad} V(X) \neq 0$, then $V^{-1}(c)$ looks like a 'surface' of dimension $n-1$ near X.

If all points in $V^{-1}(c)$ are regular points, then we say that c is a *regular value* for V.

If X is a nonregular point for V, then $\operatorname{grad} V(X) = 0$, so X is a *critical point* for the function V, since all partial derivatives of V vanish at X.

In the case $n = 2$, $V^{-1}(c)$ is a simple curve through X when X is a regular point. And if c is a regular value, then the level set $V^{-1}(c)$ is a union of simple (or nonintersecting) curves.

Suppose that Y is a vector that is tangent to the level surface $V^{-1}(c)$ at X. Then we can find a curve $\gamma(t)$ in this level set for which $\gamma'(0) = Y$. Since V is constant along γ, it follows that

$$DV_X(Y) = \left.\frac{d}{dt}\right|_{t=0} V \circ \gamma(t) = 0.$$

Suppose that Y is a vector that is tangent to the level surface $V^{-1}(c)$ at X. Then we can find a curve $\gamma(t)$ in this level set for which $\gamma'(0) = Y$. Since V is constant along γ, it follows that

$$DV_X(Y) = \left.\frac{d}{dt}\right|_{t=0} V \circ \gamma(t) = 0.$$

Thus, we have $\text{grad}V(X) \cdot Y = 0$, or, in other words, grad $V(X)$ is perpendicular to every tangent vector to the level set $V^{-1}(c)$ at X. That is, the vector field grad $V(X)$ is perpendicular to the level surfaces $V^{-1}(c)$ at all regular points of V.

Theorem (Properties of Gradient Systems)

For the system $X' = -\text{grad}\, V(X)$, the following holds:

1. *If c is a regular value of V, then the vector field is perpendicular to the level set $V^{-1}(c)$.*
2. *The critical points of V are the equilibrium points of the system.*
3. *If a critical point is an isolated minimum of V, then this point is an asymptotically stable equilibrium point.*

Example, for $n = 2$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be the function $V(x, y) = x^2(x-1)^2 + y^2$.

Example, for $n = 2$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be the function $V(x, y) = x^2(x-1)^2 + y^2$.

Then the gradient system, for $X = (x, y)^T$,

$$X' = F(X) = -\text{grad } V(X)$$

is given by

$$\begin{cases} x' & = & -2x(x-1)(2x-1) \\ y' & = & -2y. \end{cases}$$

Example, for $n = 2$

The system

$$\begin{cases} x' &= -2x(x-1)(2x-1) \\ y' &= -2y, \end{cases}$$

has three equilibrium points: $(0,0)$, $(\frac{1}{2},0)$ and $(1,0)$. The linearization at these three points yield the following matrices:

$$DF(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad DF(\frac{1}{2},0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

$$DF(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence $(0,0)$ and $(1,0)$ are sinks, while $(\frac{1}{2},0)$ is a saddle.

Example, for $n = 2$

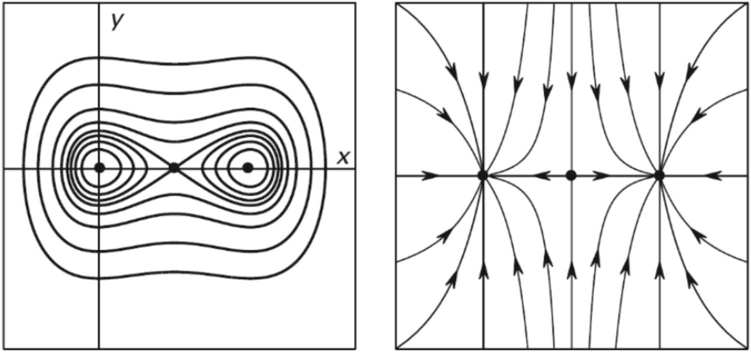
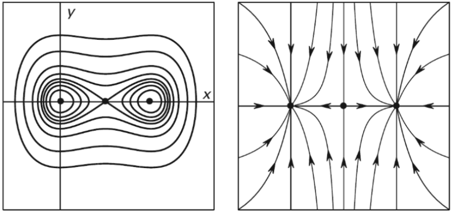


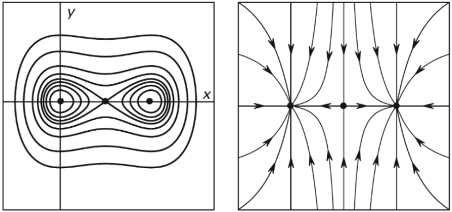
Figure: The level sets and phase portrait for the gradient system determined by $V(x, y) = x^2(x-1)^2 + y^2$.

Example, for $n = 2$



Other observations:

Example, for $n = 2$



Other observations:

- Both the x- and y-axes are invariant, as are the lines $x = \frac{1}{2}$ and $x = 1$.
- The stable curve at $(\frac{1}{2}, 0)$ is the line $x = \frac{1}{2}$.
- The unstable curve at $(\frac{1}{2}, 0)$ is the interval $(0, 1)$ on the x-axis.

Passage

A lot of gradient systems can be understood quite well.

Passage

A lot of gradient systems can be understood quite well.

Examples of gradient systems are the Cahn-Hilliard equation, the extended Fisher-Kolmogorov equation and the Swift-Hohenberg equation.

Cahn-Hilliard equation

The Cahn-Hilliard equation is given, in general, by

$$\begin{aligned}\partial_t u &= \Delta(-\Delta u + F'(u)) \\ &= -\nabla^2(\nabla^2 u - F'(u)) \\ \frac{\partial u}{\partial t} &= -\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2} - F'(u)\right),\end{aligned}$$

where $u = u(x,t)$, $x \in \Omega \in \mathbb{R}^n$ and F is a smooth function having two degenerate minima, e.g.,

$$F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2, \quad F'(u) = u^3 - u.$$

Cahn-Hilliard equation

The Cahn-Hilliard equation is given, in general, by

$$\begin{aligned}\partial_t u &= \Delta(-\Delta u + F'(u)) \\ &= -\nabla^2(\nabla^2 u - F'(u)) \\ \frac{\partial u}{\partial t} &= -\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2} - F'(u)\right),\end{aligned}$$

where $u = u(x,t)$, $x \in \Omega \in \mathbb{R}^n$ and F is a smooth function having two degenerate minima, e.g.,

$$F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2, \quad F'(u) = u^3 - u.$$

The function F is called the *potential*.

Applications

The Cahn-Hilliard equation (after John W. Cahn and John E. Hilliard) describes phase separation in binary alloys: Spinodal decomposition.

Applications

The Cahn-Hilliard equation (after John W. Cahn and John E. Hilliard) describes phase separation in binary alloys: Spinodal decomposition.

Definition (Spinodal decomposition)

When binary alloys are cooled rapidly to low temperatures below the critical point, they tend to form quickly inhomogeneities forming a granular structure.

Applications

The Cahn-Hilliard equation (after John W. Cahn and John E. Hilliard) describes phase separation in binary alloys: Spinodal decomposition.

Definition (Spinodal decomposition)

When binary alloys are cooled rapidly to low temperatures below the critical point, they tend to form quickly inhomogeneities forming a granular structure.

$\rightsquigarrow$ PATTERNS

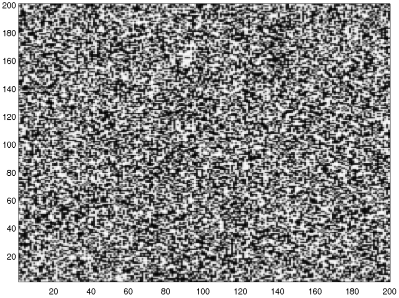


Figure: Microstructural evolution under the Cahn-Hilliard equation, demonstrating distinctive coarsening and phase separation.

Many more applications

There are many more applications of the CH-equation:

- Electric voltage
- Reacting chemicals

Many more applications

There are many more applications of the CH-equation:

- Electric voltage
- Reacting chemicals

For my masterthesis: Patterns in musselbeds.

Connection between CH-equation and gradient systems

We introduce the functional

$$W(u) = \int_\Omega \left\{ F(u) + \frac{1}{2}|\nabla^2 u|^2 \right\} \, dx,$$

where the function $F(u)$, as before, is smooth with two degenerate minima.

Connection between CH-equation and gradient systems

We introduce the functional

$$W(u) = \int_\Omega \left\{ F(u) + \frac{1}{2}|\nabla^2 u|^2 \right\} \, dx,$$

where the function $F(u)$, as before, is smooth with two degenerate minima.

The function $F(u)$ is a so-called *double well potential.*

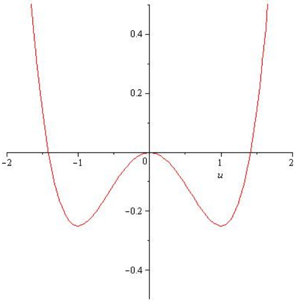


Figure: The double well potential $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$.

Connection between CH-equation and gradient systems

One can show:

$$\boxed{\frac{\partial u}{\partial t} = -K \text{grad } W(u) = -K\nabla^2(\nabla^2 u - F'(u)),}$$

where K is some positive constant or function. [1]
Here, the notion of Hilbert space is needed!

[1] [Fife]

Connection between CH-equation and gradient systems

One can show:

$$\boxed{\frac{\partial u}{\partial t} = -K \text{grad } W(u) = -K\nabla^2(\nabla^2 u - F'(u)),}$$

where K is some positive constant or function. [1]
Here, the notion of Hilbert space is needed!

Hence the Cahn-Hilliard equation is a gradient system, and W a Lyapunov function.

[1] [Fife]

Connection between CH-equation and gradient systems

One can show:

$$\boxed{\frac{\partial u}{\partial t} = -K \text{grad } W(u) = -K\nabla^2(\nabla^2 u - F'(u)),}$$

where K is some positive constant or function. [1]
Here, the notion of Hilbert space is needed!

Hence the Cahn-Hilliard equation is a gradient system, and W a Lyapunov function.

Remark: This is a very simple explanation of the CH-equation as a gradient system.

[1] [Fife]

Other equations

The Cahn-Hilliard equation, for $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$, is

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^2 u}{\partial x^2} + u - u^3\right\}.$$

Other equations

The Cahn-Hilliard equation, for $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$, is

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^2 u}{\partial x^2} + u - u^3\right\}.$$

Another fourth order parabolic differential equation, for $f(u) = u - u^3$:

$$\frac{\partial u}{\partial t} = -\gamma\frac{\partial^4 u}{\partial x^4} + \beta\frac{\partial^2 u}{\partial x^2} + f(u),$$

where $\gamma > 0$ and $\beta \in \mathbb{R}$.

Other equations

The Cahn-Hilliard equation, for $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$, is

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^2 u}{\partial x^2} + u - u^3\right\}.$$

Another fourth order parabolic differential equation, for $f(u) = u - u^3$:

$$\frac{\partial u}{\partial t} = -\gamma\frac{\partial^4 u}{\partial x^4} + \beta\frac{\partial^2 u}{\partial x^2} + f(u),$$

where $\gamma > 0$ and $\beta \in \mathbb{R}$.

$\beta > 0$: *Extended Fisher-Kolmogorov* equation (EFK),

$\beta < 0$: *Swift-Hohenberg* equation (SH).

Other equations

The Cahn-Hilliard equation, for $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$, is

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^2 u}{\partial x^2} + u - u^3\right\}.$$

Another fourth order parabolic differential equation, for $f(u) = u - u^3$:

$$\frac{\partial u}{\partial t} = -\gamma\frac{\partial^4 u}{\partial x^4} + \beta\frac{\partial^2 u}{\partial x^2} + f(u),$$

where $\gamma > 0$ and $\beta \in \mathbb{R}$.

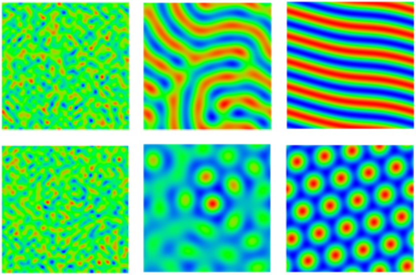
$\beta > 0$: *Extended Fisher-Kolmogorov* equation (EFK),

$\beta < 0$: *Swift-Hohenberg* equation (SH).

Note that parameters γ and β can be combined into a single parameter via a scaling of the spatial coordinate.

The EFK- and SH-equation, for various nonlinearities $f(u)$, again serve as a model in many applications:

- pattern formation in a variety of complex fluids and biological materials
- travelling water waves in a shallow channel.



The Fisher-Kolmogorov equation

A simular, more simple equation, for $\beta > 0$ and $\gamma = 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3.$$

The *Fisher-Kolmogorov* equation (FK).

The Fisher-Kolmogorov equation

A simular, more simple equation, for $\beta > 0$ and $\gamma = 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3.$$

The *Fisher-Kolmogorov* equation (FK).

Nonlinear reaction-diffusion equation, which is extensively studied.

The stationary solutions of the FK-equation satisfy the ODE:

$$u'' = -u + u^3. \tag{1}$$

The stationary solutions of the FK-equation satisfy the ODE:

$$u'' = -u + u^3. \tag{1}$$

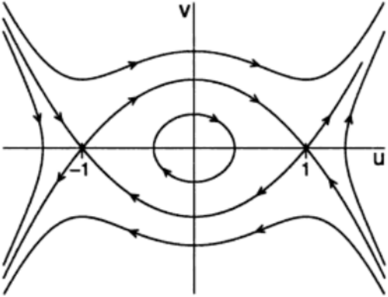


Figure: The phase plane of (1) in the $(u, v) = (u, u')$-plane.

The stationary solutions of the FK-equation satisfy the ODE:

$$u'' = -u + u^3. \tag{1}$$

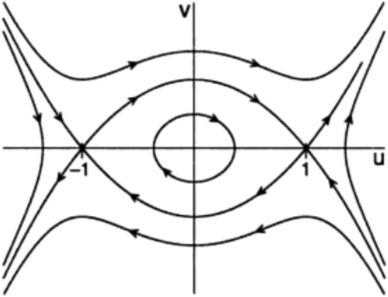
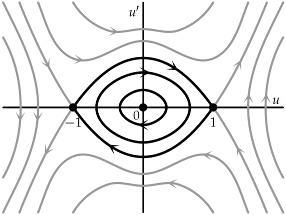


Figure: The phase plane of (1) in the $(u, v) = (u, u')$-plane.

Bounded solutions of the FK-equation:

- Constant solutions: $u(x) \equiv 0$ (unstable), $u(x) \equiv 1$, $u(x) \equiv -1$ (stable).
- Two kinks or heteroclinic solutions connecting $(u, u') = (\pm 1, 0)$: $u(x) = \pm \tanh\left(\frac{x}{\sqrt{2}}\right)$.
- Periodic solutions: Infinitely many solutions, which oscillate around $u = 0$.



Introduce the energy functional or Hamiltonian:

$$E(u) = \frac{1}{2}(u')^2 - \frac{1}{4}(u^2 - 1)^2,$$

which is constant along solutions of (1).

$\leadsto$ The classical energy of a particle in a potential.

Introduce the energy functional or Hamiltonian:

$$E(u) = \frac{1}{2}(u')^2 - \frac{1}{4}(u^2 - 1)^2,$$

which is constant along solutions of (1).

$\leftrightsquigarrow$ The classical energy of a particle in a potential.

In connection with the Hamiltonian: an action functional, *Lagrangian action*:

$$J(u) = \int \left(\frac{1}{2}(u')^2 + \frac{1}{4}(1 - u^2)^2 \right) \, dx.$$

Here $J(u)$ is a Lyapunov function for the flow of the original FK-equation.

The EFK- and SH-equation

The Extended Fisher-Kolmogorov equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \beta > 0.$$

The EFK- and SH-equation

The Extended Fisher-Kolmogorov equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \beta > 0.$$

The Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2 u}{\partial x^2}\right)^2 + \alpha u - u^3, \quad \alpha \in \mathbb{R}$$

can be rescaled to

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + u - u^3,$$

with $\beta = -\frac{2}{\sqrt{\alpha - 1}} < 0$.

General equation

Equations of the general form

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + f(u), \quad \gamma > 0, \beta \in \mathbb{R},$$

where $f(u)$ is a nonlinear function.

General equation

Equations of the general form

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + f(u), \quad \gamma > 0, \beta \in \mathbb{R},$$

where $f(u)$ is a nonlinear function.

For example,

$$f(u) = u - u^3, \quad \text{and therefore } F(u) = \int f(s)\, ds = \frac{1}{2}u^2 - \frac{1}{4}u^4.$$

General equation

Again interested in the stationary (time-independent) solutions:

$$-\gamma u'''' + \beta u'' + f(u) = 0,$$

where we set $f(u) = u - u^3$.

General equation

Again interested in the stationary (time-independent) solutions:

$$-\gamma u'''' + \beta u'' + f(u) = 0,$$

where we set $f(u) = u - u^3$.

The energy functional or Hamiltonian is

$$E(u) = -\gamma u' u''' + \frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 + F(u).$$

Here $F(u)$ is the potential.

General equation

Again interested in the stationary (time-independent) solutions:

$$-\gamma u'''' + \beta u'' + f(u) = 0,$$

where we set $f(u) = u - u^3$.

The energy functional or Hamiltonian is

$$E(u) = -\gamma u' u''' + \frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 + F(u).$$

Here $F(u)$ is the potential.

The Lagrangian action associated with this Hamiltonian is

$$J(u) = \int \left(\frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 - F(u) \right) \, dx.$$

Here $J(u)$ is a Lyapunov function for the flow of the original general form of the EFK-equation.

Comparison

The functional for the Cahn-Hilliard equation:

$$W(u) = \int \left\{ F(u) + \frac{1}{2}|\nabla^2 u|^2 \right\} \, dx.$$

The functional for the general stationary equation:

$$J(u) = \int \left(\frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 - F(u)) \right) \, dx.$$

Comparison

The functional for the Cahn-Hilliard equation:

$$W(u) = \int \left\{ F(u) + \frac{1}{2}|\nabla^2 u|^2 \right\} \, dx.$$

The functional for the general stationary equation:

$$J(u) = \int \left(\frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 - F(u) \right) \, dx.$$

Both Lyapunov functions!

Last remarks

A lot of research has been done for these type of equations.

Last remarks

A lot of research has been done for these type of equations.

Goal for my master thesis:
To describe the patterns found in musselbeds, using the Cahn-Hilliard equation.

Thank you for your attention!