# Cracking a non–homogeneous reaction–diffusion system: A root hair plant initiation model

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Seminar on Spatio–Temporal Patterns
Utrecht University, 18th April 2012

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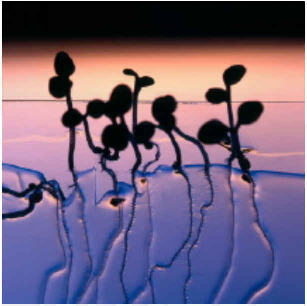

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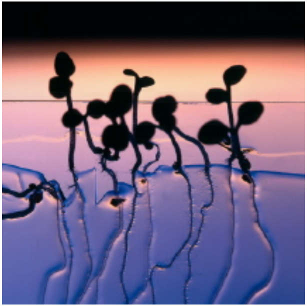

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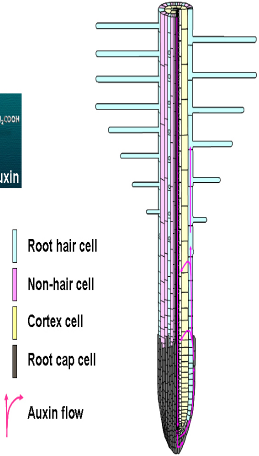
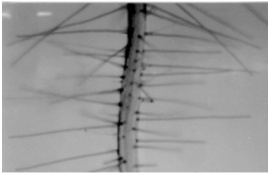
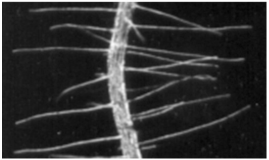
- Morphogenesis of plants.
- Physical and chemical interactions.
- Root hairs of plants as biological model.
- Important role of auxin.

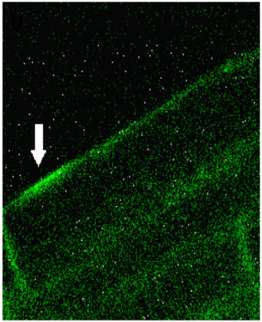



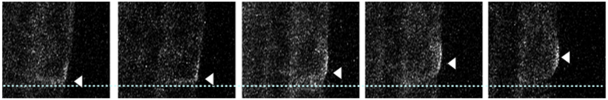

J. D. Jones, 2011

## Seeking for motives

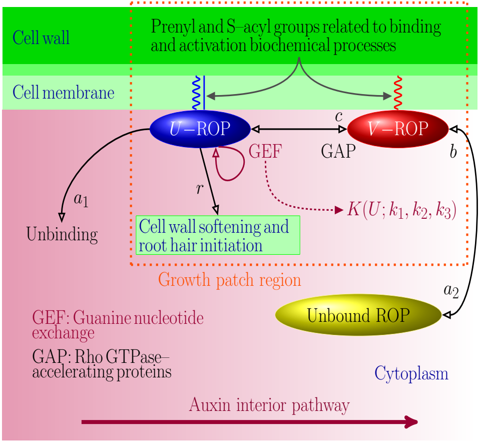








- ☞ How do these interactions occur and, consequently, trigger outgrowth?
- ☞ How is this growth governed such that it happens at specific times and places?
- ☞ Can we do educated guesses leading to the better understanding of these interactions and experiments designing?
- ☞ Can we predict—analytically—patch location and root hair phenotype conditions to be occurred?




Activation and binding processes catalysed by *auxin* are represented by

$$K(U; k_i) = k_1 + \frac{k_2 U^q}{1 + k_3 U^q},$$

$$q > 0\ (= 2)$$

$$k_2 = \begin{cases} k_{20}; & \text{no gradient} \\ k_{20}\alpha(\mathbf{x}); & \text{gradient} \end{cases}$$

## The original problem

$$\text{Active ROP:} \qquad U_t = D_1 \Delta U + K(U; k_i)V - (c + r)U,$$

$$\text{in} \quad \mathbf{x} \in \Omega, \; t > 0$$

$$\text{Inactive ROP:} \qquad V_t = D_2 \Delta V - K(U; k_i)V + cU + b,$$

where

$$K(U; k_i) = k_1 + k_2 U^2,$$

and

$$\frac{\partial}{\partial \mathbf{n}} \begin{bmatrix} U \\ V \end{bmatrix} = \mathbf{0} \qquad \text{in} \quad \partial\Omega.$$

## The fundamental system

$$\begin{cases} U_t = \delta U_{xx} + \alpha(x)U^2V - U + \frac{1}{\tau\gamma}V, & \text{in} \quad 0 < x < 1,\ t > 0 \\ \tau V_t = DV_{xx} - V + 1 - \tau\gamma\left[\alpha(x)U^2V - U\right] - \beta\gamma U, \end{cases}$$

where

$$\delta := \frac{D_1}{L^2(c+r)}, \quad D := \frac{D_2}{L^2 k_1}, \quad \tau := \frac{c+r}{k_1}, \quad \alpha_0 := \frac{k_{20}}{c+r},$$

$$\gamma := \frac{k_1^2}{\alpha_0 b^2}, \qquad \beta := \frac{r}{k_1}$$

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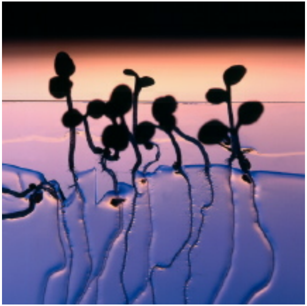

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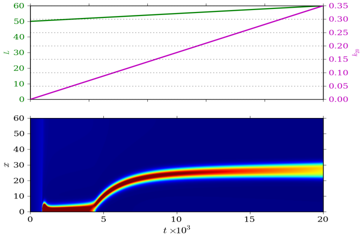
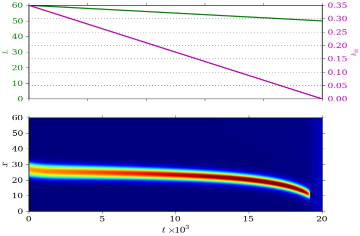
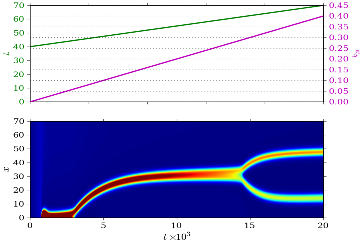
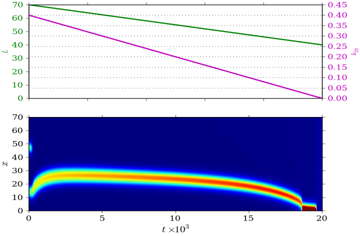
## Lengthening & auxin sweeping



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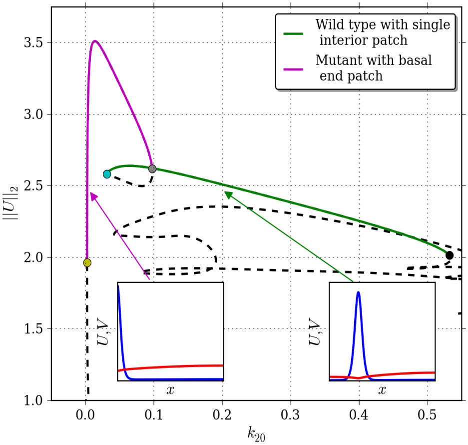
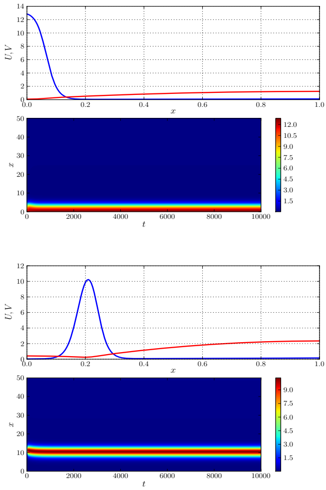
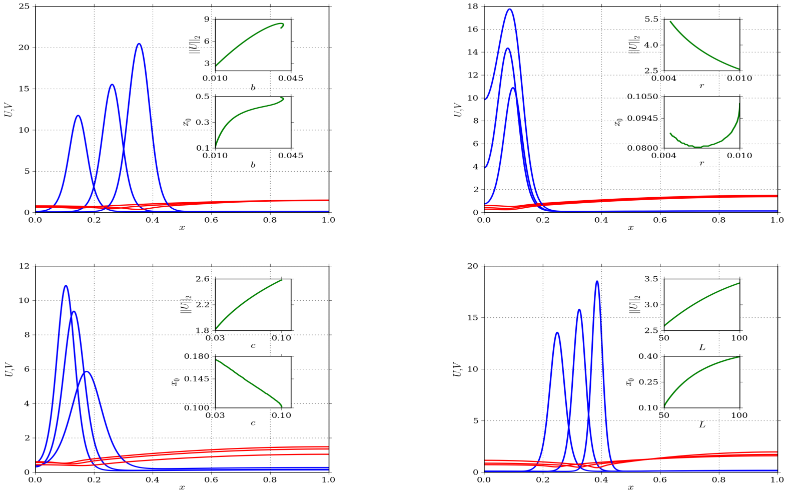
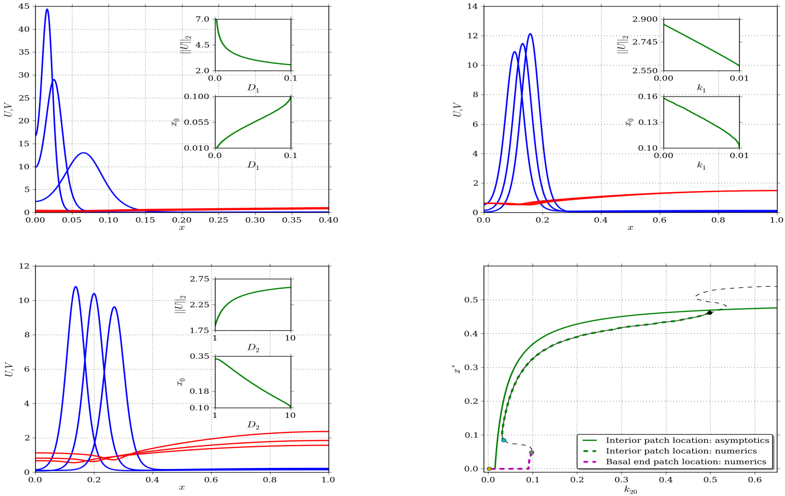


## Chasing solutions



## Chasing solutions

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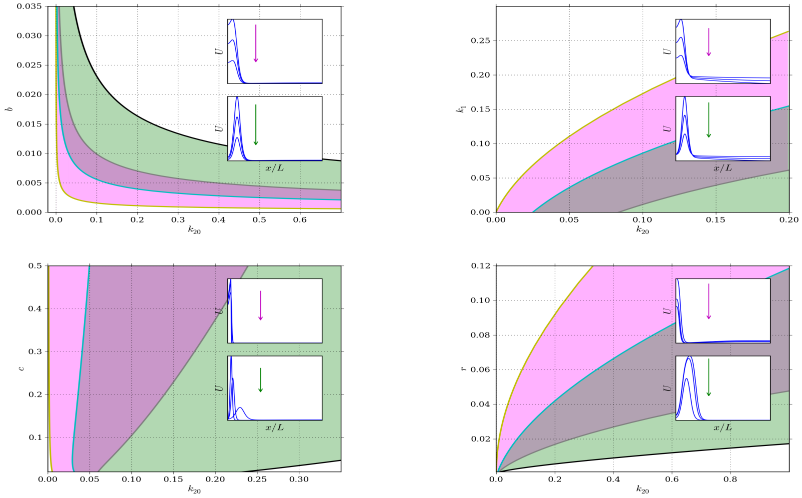
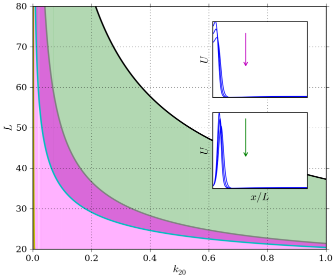
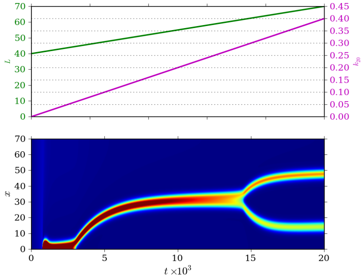
## Chasing solutions



## Chasing solutions



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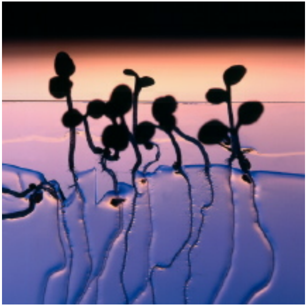

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$$\varepsilon := \sqrt{\delta}, \qquad U = \varepsilon^{-1} u, \quad V = \varepsilon v, \quad D = \varepsilon^{-1} D_0$$

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## Outer scale system

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + \alpha(x) u^2 v - u + \dfrac{\varepsilon^2}{\tau\gamma} v, \\ \varepsilon\tau v_t = D_0 v_{xx} + 1 - \varepsilon v - \varepsilon^{-1}\left(\tau\gamma\left(\alpha(x) u^2 v - u\right) + \beta\gamma u\right). \end{cases}$$

$$x_0 = x_0(\eta), \qquad \eta = \varepsilon^2 t, \quad \xi = \varepsilon^{-1}(x - x_0),$$

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## Inner scale system

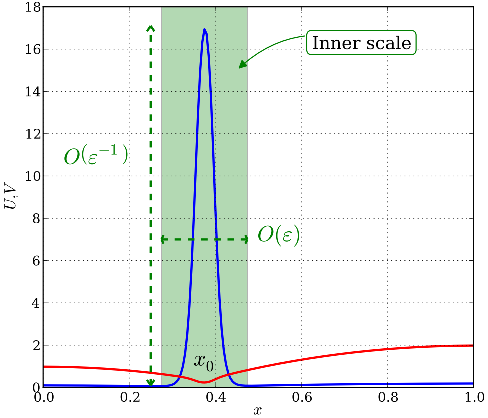
$$\begin{cases} -\varepsilon^{-1}\dfrac{d\eta}{dt}\dfrac{dx_0}{d\eta}u_\xi & = & u_{\xi\xi} + (\alpha(x_0) + \varepsilon\alpha'(x_0)\xi)\, u^2 v - u + \dfrac{\varepsilon^2}{\tau\gamma}v, \\ -\varepsilon^2\tau\dfrac{d\eta}{dt}\dfrac{dx_0}{d\eta}v_\xi & = & D_0 v_{\xi\xi} - \varepsilon\tau\gamma\left((\alpha(x_0) + \varepsilon\alpha'(x_0)\xi)\, u^2 v - u\right) - \\ & & -\varepsilon\beta\gamma u + \varepsilon^2 + \varepsilon^3 v. \end{cases}$$

$$U(x) \sim \frac{\varepsilon^{-1}}{6\beta\gamma} w\left(\varepsilon^{-1}(x - x_0)\right),$$

$$V(x_0) \sim \varepsilon \frac{6\beta\gamma}{\alpha(x_0)}$$

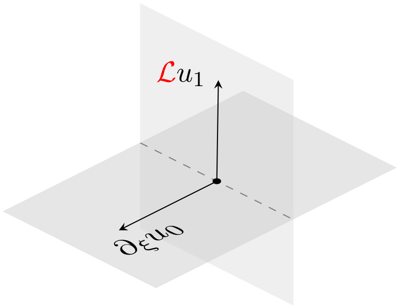
$$U(x) \sim \frac{\varepsilon^2}{\tau\gamma} v_0(x),$$

$$V(x) \sim \varepsilon v_0(x),$$




$$v_0(x) = \frac{6\beta\gamma}{\alpha(x_0)} + \frac{1}{D_0}\left(G(x; x_0) - G(x_0; x_0)\right)$$

$$-\frac{dx_0}{d\eta}\partial_\xi u_0 = \underbrace{\partial_{\xi\xi}u_1 - u_1 + 2\alpha(x_0)v_0u_0u_1}_{\mathcal{L}u_1} + \left(\alpha(x_0)v_1 + \alpha'(x_0)\xi v_0\right)u_0^2,$$




where

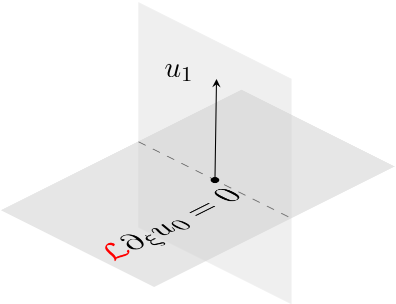
$$v_1(\xi) = v\left(\varepsilon^{-1}(x - x_0)\right) - v^0.$$

$$\frac{D_0}{\tau\gamma}\partial_{\xi\xi}v_1 - \alpha(x_0)v_0u_0^2 + u_0 - \frac{\beta}{\tau}u_0 = 0.$$

## Second order

$$-\frac{dx_0}{d\eta} = \frac{1}{3v^0}\frac{\langle v_1, \partial_\xi w^3\rangle}{\|\partial_\xi w\|^2} + \frac{\alpha'(x_0)}{3\alpha(x_0)}\frac{\langle \xi, \partial_\xi w^3\rangle}{\|\partial_\xi w\|^2},$$

where




$$u_0 = \frac{w}{\alpha(x_0)v^0}.$$

$$D_0\left(\partial_\xi v_1(\infty) - \partial_\xi v_1(-\infty)\right) = \frac{6\beta\gamma}{\alpha(x_0)v^0} = 1.$$

Matching condition:

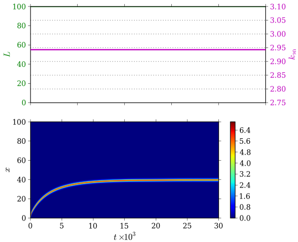
$$\partial_\xi v_1(\pm\infty) = \frac{1}{D_0} G_x\left(x_0^{\pm}; x_0\right)$$

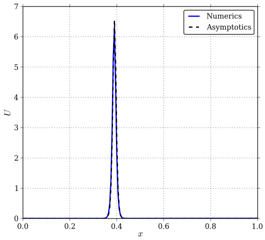
**Result (Single interior spike location dynamics)**

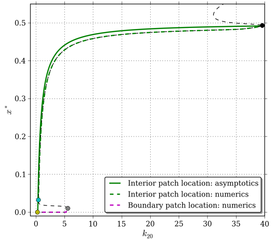
*For $\varepsilon \ll 1$ and $D \sim O\left(\varepsilon^{-1}\right)$, let $\eta = \varepsilon^2 t$. Then the spike position $x_0(\eta)$ of the slow dynamics is described by*

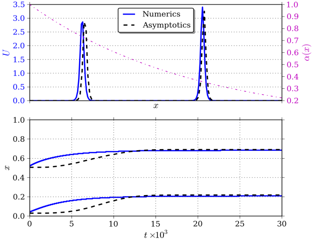
$$\frac{dx_0}{d\eta} = \frac{1}{3\beta\gamma D_0}\alpha(x_0)\left(\frac{1}{2} - x_0\right) + 2\frac{\alpha'(x_0)}{\alpha(x_0)}$$

## Extensions




$$U(x) \sim \frac{\varepsilon^{-1}}{6\beta\gamma} \sum_{j=0}^{N} w(\xi_j)\, n_j, \qquad \sum_{j=0}^{N} n_j = 1,$$

$$n_j := \frac{6\beta\gamma}{\alpha(x_j) v^j}, \qquad v^j := v(x_j)$$

## Result (Two interior spike location dynamics)

*For $\varepsilon \ll 1$, the location dynamics of the two–spikes case is described by*

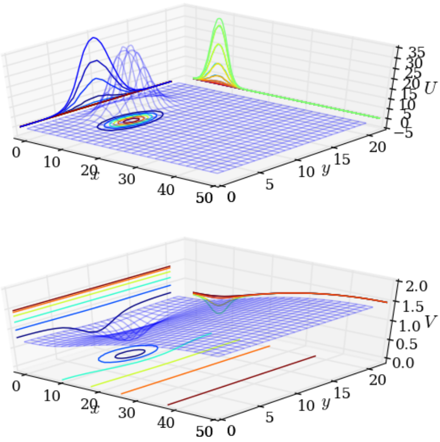
$$\frac{dx_0}{d\eta} = \frac{n_0}{3\beta\gamma D_0}\alpha(x_0)\left[n_0\left(\frac{1}{2} - x_0\right) - (1-n_0)\,x_0\right] + 2\frac{\alpha'(x_0)}{\alpha(x_0)},$$

$$\frac{dx_1}{d\eta} = \frac{1-n_0}{3\beta\gamma D_0}\alpha(x_1)\left[(1-n_0)\left(\frac{1}{2} - x_1\right) + n_0(1-x_1)\right] + 2\frac{\alpha'(x_1)}{\alpha(x_1)},$$

*where $n_0$ satisfies equation*

$$\pi_2\left(n_0; x_0, x_1\right) = \rho_2\left(n_0; x_0, x_1\right).$$

## Extensions



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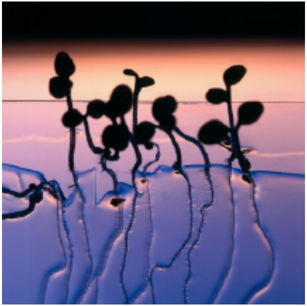

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1. Proceed in the usual way

$$u(t,x) = u_s + e^{\lambda t}\varphi(x), \quad v(t,x) = v_s + e^{\lambda t}\psi(x), \quad \varphi,\psi \ll 1.$$

2. Look for an eigenfunction in the form

$$\varphi(x) \sim \sum_{j=0}^{N} \varphi_j\left(\varepsilon^{-1}(x - x_j)\right), \quad \varphi_j \longrightarrow 0 \quad \text{as} \quad |\xi| \longrightarrow \infty.$$

3. Approximate singular terms asymptotically by a Dirac–δ function.
4. Obtain the NLBVP for vector $\mathbf{\Phi} = \mathbf{Q}\left[\varphi_0, \ldots, \varphi_N\right]^T$,

## Multiple spikes

$$\boldsymbol{\Phi}_{\xi\xi} - \boldsymbol{\Phi} + 2w\boldsymbol{\Phi} - \boldsymbol{\theta}_\lambda w^2 \left( \frac{\int_{-\infty}^{\infty} w\boldsymbol{\Phi}\, d\xi}{\int_{-\infty}^{\infty} w^2\, d\xi} \right) = \lambda\boldsymbol{\Phi}, \quad -\infty < \xi < \infty,$$

$$\boldsymbol{\theta}_\lambda = \operatorname{diag}\left(\theta_j(\lambda)\right), \quad \theta_j(\lambda) = \mu_j \frac{\lambda + 1 - 2\kappa}{\lambda + 1 - \mu_j\kappa},$$

$$\|\boldsymbol{\Phi}\| \longrightarrow 0 \quad \text{as} \quad |\xi| \longrightarrow \infty.$$

## Result (ROP competition stability)

*The quasi–equilibrium solution of the* OUTER SCALE SYSTEM *with spikes at* $x_0, \ldots, x_N$ *is unstable on an* $O(1)$ *time–scale if there exists at least one* $j = 0, \ldots, N$ $(N \geq 1)$ *for which*

$$\sigma_j > \sigma^*, \quad \sigma^* := \frac{1}{6\beta D_0 \gamma},$$

*where the eigenvalues of the NLBVP satisfy* $\lambda > -\beta/\tau$ *and* $\lambda > -(\sigma_j + \sigma^*)$.

## Corollary

*For* $\Lambda := \kappa L^3 k_{20}$*, instability on an* $O(1)$ *time–scale is presented if*

*Two interior spikes:*

$$\Lambda > \Lambda^* = \frac{1}{6\beta l_0}\left[\frac{1}{\alpha(x_0)n_0^2} + \frac{1}{\alpha(x_1)\left(1-n_0\right)^2}\right]^{-1}, \quad l_0 = \frac{1}{x_1 - x_0}$$

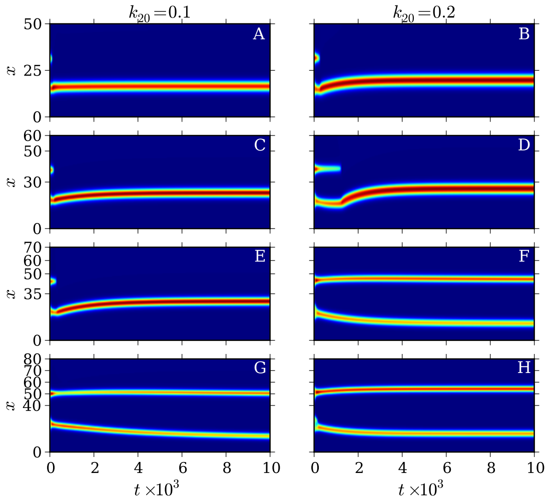
*Boundary and interior spikes:*

$$\Lambda > \Lambda^* = \frac{1}{6\beta l_1}\left[\frac{1}{4\alpha(0)\left(1-n_1\right)^2} + \frac{1}{\alpha(x_0)n_1^2}\right]^{-1}, \quad l_1 = \frac{1}{x_0}$$

An alternating–sign–fluctuation of spike amplitude is given by

$$\mathbf{y}_1 = [1, -1]^T$$

## Illustrations



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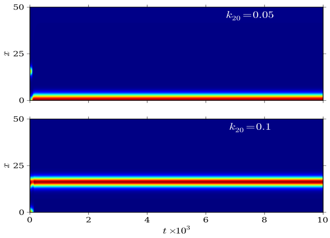


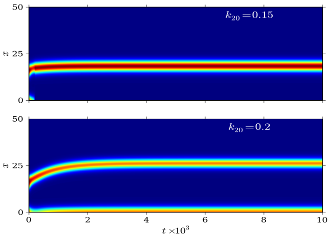


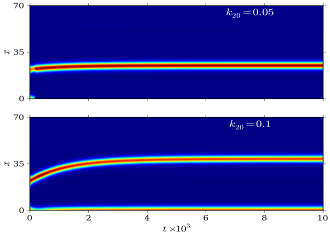


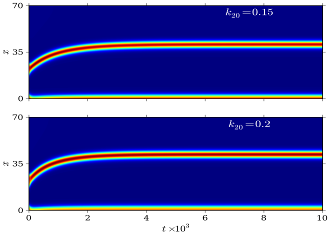

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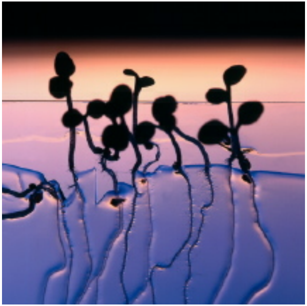

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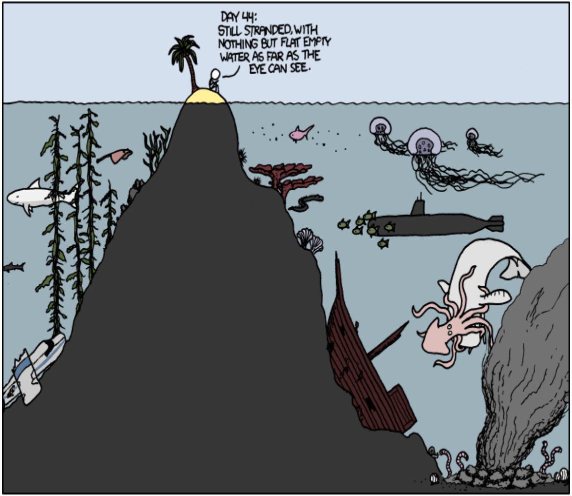
- Spike formation is a direct consequence of bistability. **Autocatalysis governs active–ROPs aggregation.**
- Spike position of the slow dynamics is described theoretically. **The gradient controls the location of the patch.**
- Turing pattern is destroyed by spatial inhomogeneity, providing robust wave–pinned–like solutions. **Robustness is particularly relevant to model theoretically biological interactions.**
- Early multiple spikes configuration can be killed by some instability at fast time scale. **This could supply theoretical highlights for mutants to be occurred.**




R. Munroe, 2010

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