Bifurcation Analysis of DDEs

Initial-value problems.
Stability and continuation of equilibria and limit cycles.

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Contents

Literature

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1. Initial-value problems for DDEs with constant delays

Consider a **DDE** for $x(t) \in \mathbb{R}^n$:

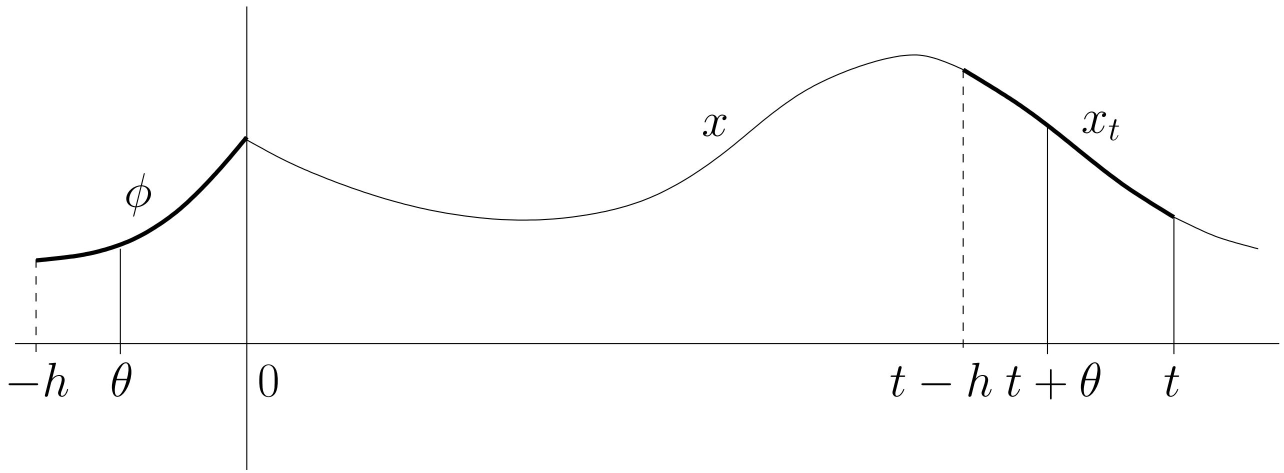
$$\dot{x}(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m)),$$

where $f : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ is smooth, and

$$0 =: \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m =: h < \infty,$$

with **initial data** $\phi \in C([-h, 0], \mathbb{R}^n)$.

- **Global solution**: $x \in C([-h, \infty), \mathbb{R}^n) \cap C^1([0, \infty), \mathbb{R}^n)$.
- **History** for $t \geq 0$: $x_t \in C([-h, 0], \mathbb{R}^n)$, $\quad x_t(\theta) := x(t+\theta), \quad \theta \in [-h, 0]$.



- **Initial-value problem** for DDE:

$$\begin{cases} \dot{x}(t) &= F(x_t), \quad t \geq 0, \\ x_0 &= \phi, \end{cases}$$

where $F : C([-h,0],\mathbb{R}^n) \to \mathbb{R}^n$ is smooth and is defined by

$$F(\phi) = f(\phi(0), \phi(-\tau_1), \phi(-\tau_2), \ldots, \phi(-\tau_m)).$$

- If F is globally Lipschitz, then for any $\phi \in C([-h,0],\mathbb{R}^n)$ there exists a unique global solution $x = x(\cdot,\phi)$ for the above IVP that depends continuously on ϕ on any interval $[0,T]$. If F is only locally Lipschitz, the solution is only guaranteed to exist on a small interval.

- The IVP defines a (local) **semigroup** $S(t)$ on $X = C([-h,0],\mathbb{R}^n)$

$$[S(t)(\phi)](\theta) := x_t(\theta), \quad \theta \in [-h,0],\ t \geq 0,$$

which is **strongly continuous**, i.e.

$$\lim_{t \downarrow 0} \|S(t)\phi - \phi\| = 0, \quad \phi \in X.$$

2. Numerical solution of IVPs

- Simplest approach: Use an explicit ODE solver with interpolation of the history x_t known only at mesh points.

- MATLAB **dde23** function: Runge-Kutta BS(2,3) with cubic Hermite interpolation x_H between consequtive mesh points. Let $h_k < \tau_1$ be the current **stepsize**. For $i = 1, 2, 3$:

$$
\begin{aligned}
t_{ki} &= t_k + c_i h_k, \\
f_{ki} &= f(x_{ki}, x_H(t_{ki} - \tau_1), x_H(t_{ki} - \tau_2), \ldots, x_H(t_{ki} - \tau_m)), \\
&\quad \text{where } x_{ki} = x_k + h_k \sum_{j=1}^{i-1} a_{ij} f_{kj}.
\end{aligned}
$$

Then $t_{k+1} = t_k + h_k$ and

$$
x_{k+1} = x_k + h_k \sum_{i=1}^{3} b_i f_{ki} + O(h_k^4), \quad \tilde{x}_{k+1} = x_k + h_k \sum_{i=1}^{3} \tilde{b}_i f_{ki} + O(h_k^3).
$$

- Use $\|x_{k+1} - \tilde{x}_{k+1}\|$ to adapt the stepsize h_k.

3. Equilibria of DDEs and their stability

- **Equilibrium** (constant) solution $x(t) = x^* \in \mathbb{R}^n$:

$$f(x^*, x^*, x^*, \ldots, x^*) = 0.$$

- Linearized DDE:

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{m} A_j y(t-\tau_j), \;\; A_j = D_j f(x^*, x^*, x^*, \ldots, x^*), \; j = 0, 1, \ldots, m.$$

- **Characteristic matrix**:

$$\Delta(\lambda) := \lambda I_n - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \tau_j}, \quad \lambda \in \mathbb{C}.$$

- The **characteristic equation**

$$\det \Delta(\lambda) = 0$$

has an infinite number of roots. The equilibrium x^* is **asymptotically stable** if all characteristic roots satisfy $\Re(\lambda) < 0$.

- Let $T(t)$ be the strongly continuous semigroup corresponding to the linearized DDE. It holds: $T(t) = D_\phi(S(t)(\phi))\Big|_{\phi=x^*}$.
  The **infinitesimal generator** of $T(t)$

$$A\phi := \lim_{t\downarrow 0} \frac{1}{t}(T(t)\phi - \phi)$$

  is given by $(A\phi)(\theta) = \dot{\phi}(\theta)$ for $\phi \in D(A)$ where

$$D(A) = \{\phi \in X : \dot{\phi} \in X \quad \text{and} \quad \dot{\phi}(0) = \sum_{j=0}^{m} A_j \phi(-\tau_j)\}.$$

  If λ is a characteristic root, then λ belongs to the **spectrum** $\sigma(A)$ of A and $\mu = e^{\lambda\delta} \in \sigma(T(\delta))$. Thus, eigenvalues of a discretization of $T(\delta)$ can be used to approximate characteristic roots.

- If an approximation to an eigenvalue λ is known, it can be accurately computed by **Newton iterations** applied to the system

$$\begin{cases} \Delta(\lambda)v & = & 0, \\ \langle v, v_0 \rangle & = & 1, \end{cases}$$

  where $(\lambda, v) \in \mathbb{C}^{n+1}$ are unknown and $v_0 \in \mathbb{C}^n$ is fixed.

4. Computation of equilibria

Consider now a DDE depending on **parameter** $\alpha \in \mathbb{R}$:

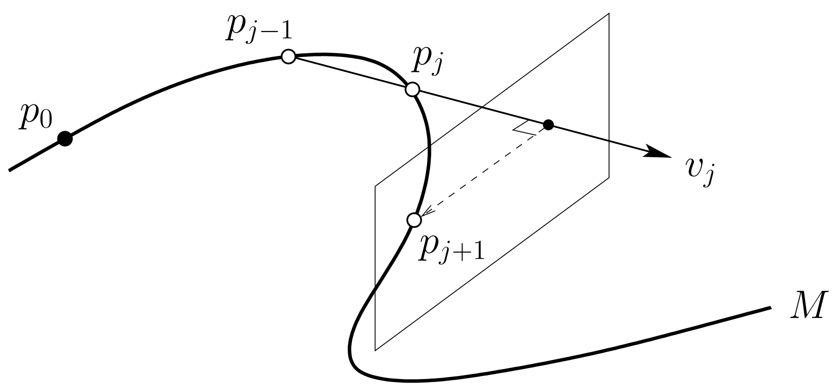
$$\dot{x}(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m), \alpha),$$

where $f : \mathbb{R}^{n(m+1)} \times \mathbb{R} \to \mathbb{R}^n$ is smooth.

- An **equilibrium manifold** M is defined by

$$G(u, \alpha) := f(u, u, u, \ldots, u, \alpha) = 0, \quad G : \mathbb{R}^{n+1} \to \mathbb{R}^n.$$

- Near any regular point $p_0 = (u_0, \alpha_0)$, the system $G(u, \alpha) = 0$ defines a unique smooth curve that passes through p_0 and can be found by **numerical continuation**. **DDE-BIFTOOL** employs a secant prediction followed by Newton corrections in the orthogonal plane.



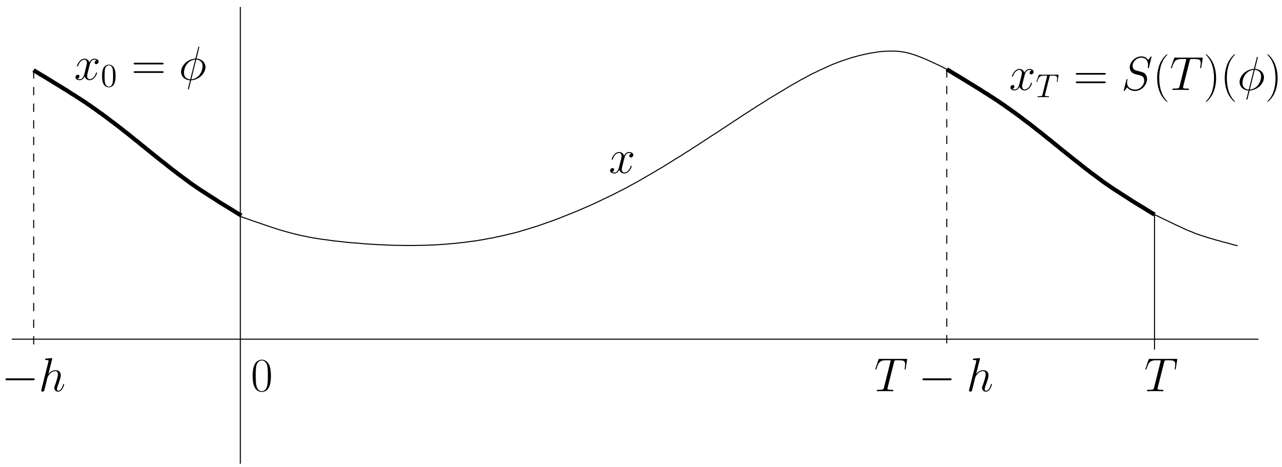
5. Cycles of DDEs and their stability

Consider first a **DDE** without parameters:

$$\dot{x}(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m)),$$

where $f : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ is smooth.

- $x(t_0 + T) = x(t_0)$ for some $t_0 \geq 0$ does not imply periodicity of the whole solution, i.e. $x(t+T) = x(t)$ for all $t \geq 0$!
- **Periodicity condition**: $S(T)(\phi) = \phi$ or $x_T = x_0$



- **Phase condition**: $\Psi[x, T] = 0$, e.g. an integral phase condition.

Stability of cycles:

- **Monodromy operator**: $Y := D_\phi(S(T)(\phi))\Big|_{\phi=x^*}$ where $x^* = x^*(t)$ is the periodic solution. If all eigenvalues of Y (**multipliers**) are located strictly inside the unit circle, the cycle is **asymptotically stable**.

- The linearized about $x^*(t)$ DDE:

$$\dot{y}(t) = A_0(t)y(t) + \sum_{j=1}^{m} A_j(t)y(t-\tau_j),$$

  where

$$A_j(t) = D_j f(x^*(t), x^*(t-\tau_1), \ldots, x^*(t-\tau_m)), \quad j = 0, 1, \ldots, m.$$

- Let $U(t,s) : X \to X$ be the **solution operator** for the linearized DDE, i.e. $y_t = U(t,s)y_s$. Then

$$Y = U(T, 0).$$

- In **DDE-BIFTOOL**, a matrix approximation to Y is computed via orthogonal collocation.

6. Computation of cycles

- Stable periodic solutions can be found by **integration**.

- **Periodic BVP**:

$$\begin{cases} \dot{x}(t) - f(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m)) & = & 0, \quad t \in [0, T] \\ x_T - x_0 & = & 0, \\ \Psi[x, T] & = & 0. \end{cases}$$

- **Rescaled periodic BVP**:

$$\begin{cases} \dot{u}(s) - Tf\left(u(s), u\left(s - \frac{\tau_1}{T}\right), u\left(s - \frac{\tau_2}{T}\right), \ldots, u\left(s - \frac{\tau_m}{T}\right)\right) & = & 0, \ s \in [0, 1] \\ u(\theta + 1) - u(\theta) & = & 0, \ \theta \in \left[-\frac{h}{T}, 0\right] \\ \psi[u] & = & 0, \end{cases}$$

where for some reference 1-periodic function $u^{(0)}$

$$\psi[u] = \int_0^1 \dot{u}^{(0)}(s)(u^{(0)}(s) - u(s))\, ds.$$

Discretization:

- **Mesh points** $0 = s_0 < s_1 < \cdots < s_L = 1$
- **Basis points** $s_{i,j} = s_i + \frac{j}{M}(s_{i+1} - s_i), i = 0, 1, \ldots, L-1, j = 1, \ldots, M-1$
- **Continuous approximation**

$$u(s) = \sum_{j=0}^{M} u^{i,j} P_{i,j}(s), \quad s \in [s_i, s_{i+1}],$$

where $P_{i,j}(s)$ are the **Lagrange basis polynomials**

$$P_{i,j}(s) = \prod_{k=0,k\neq j}^{M} \frac{s - s_{i,k}}{s_{i,j} - s_{i,k}}, \;\; j = 0, 1, \ldots, M-1.$$

- **Unknowns** $\left(\{u^{i,j}\}_{j=0,\ldots,M-1}^{i=0,1,\ldots,L-1}, u^{L,0}, T\right) \in \mathbb{R}^{n(LM+1)+1}$

Orthogonal collocation:

- **Collocation points:**

$$c_{i,j} = \tau_i + c_j(s_{i+1} - s_i), \quad i = 0, 1, \ldots, L-1, \; j = 1, \ldots, M,$$

where c_j are the roots of the M-th degree **Gauss-Legendre polynomial** transformed to $[0, 1]$.

- **Defining system** (with $n(LM+1)+1$ scalar equations)

$$\begin{cases} \dot{u}(c_{i,j}) - Tf\left(u(c_{i,j}), u\left(\left(c_{i,j} - \frac{\tau_1}{T}\right) \bmod 1\right), \ldots, u\left(\left(c_{i,j} - \frac{\tau_m}{T}\right) \bmod 1\right)\right) & = & 0, \\ u^{0,0} - u^{L,0} & = & 0, \\ \psi[u] & = & 0. \end{cases}$$

- **Approximation error:** $\|u(s_{i,j}) - u^{i,j}\| = O(\delta^M)$ where

$$\delta := \max_{i=0,1,\ldots,L-1} |s_{i+1} - s_i|$$

- If the DDE depends on **parameter** $\alpha \in \mathbb{R}$, the above defining system can be used for numerical continuation of the cycle.