Bifurcation Analysis of DDEs

Simplest local bifurcations. Critical normal forms for codim 1 bifurcations of equilibria.

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Literature


1. Simplest critical equilibria and their computation

Consider a DDE with \( m \) delays for \( x(t) \in \mathbb{R}^n \) and parameter \( \alpha \in \mathbb{R} \):

\[
\dot{x}(t) = f(x(t), x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_m), \alpha),
\]

where \( f : \mathbb{R}^{(m+1)n} \times \mathbb{R} \to \mathbb{R}^n \) is smooth and \( 0 = \tau_0 < \tau_1 < \cdots < \tau_m = h \). Assume that \( f(0) := f(0, 0, 0, \ldots, 0, 0) = 0 \), i.e. \( x = 0 \) is an equilibrium at \( \alpha = 0 \).

- Let \( \lambda_j \in \mathbb{C} \) be roots of the **characteristic equation** \( \det \Delta(\lambda) = 0 \),

\[
\Delta(\lambda) = \lambda I_n - \sum_{j=0}^{m} A_j e^{-\lambda \tau_j}, \quad A_j = D_j f(0, 0), j = 0, 1, \ldots, m.
\]

- **Codim 1 critical cases** for stability: \( \Re(\lambda) = 0 \)

\[
\begin{array}{c|c}
\mathbb{C} & \mathbb{C} \\
\hline
\lambda_1 = 0 & \lambda_1 = i\omega_0 \\
\lambda_2 = -i\omega_0 \\
\end{array}
\]
Defining systems

Let
\[ \Delta(\lambda, u, \alpha) := \lambda I_n - \sum_{j=0}^{m} \tilde{A}_j(u, \alpha)e^{-\lambda \tau_j}, \]
where \( \tilde{A}_j(u, \alpha) := D_j f(u, u, u, \ldots, u, \alpha), \) \( \tilde{A}_j(0, 0) = A_j, \) \( j = 0, 1, \ldots, m. \)

- **Fold:** \( \lambda_1 = 0 \)

\[
\begin{align*}
  f(u, u, u, \ldots, u, \alpha) &= 0, \\
  \Delta(0, u, \alpha)q &= 0, \\
  cq &= 1,
\end{align*}
\]
where \((q, u, \alpha) \in \mathbb{R}^{2n+1}, c \in \mathbb{R}^{n*}.\)

- **Andronov-Hopf:** \( \lambda_{1, 2} = \pm i\omega_0, \) \( \omega_0 > 0 \)

\[
\begin{align*}
  f(u, u, u, \ldots, u, \alpha) &= 0, \\
  \Delta(i\omega_0, u, \alpha)q &= 0, \\
  cq &= 1,
\end{align*}
\]
where \((q, u, \alpha, \omega_0) \in \mathbb{C}^n \times \mathbb{R}^n \times \mathbb{R}^2, c \in \mathbb{C}^{n*}.\)
2. Center manifold reduction

- Let $T(t)$ be the **semigroup** on $X = C([-h,0],\mathbb{R}^n)$ defined by

$$\dot{y}(t) = A_0 y(t) + \sum_{j=1}^{m} A_j y(t - \tau_j),$$

and $A$ its **infinitesimal generator**: $(A\phi)(\theta) = \dot{\phi}(\theta)$ for $\phi \in D(A)$,

$$D(A) = \{ \phi \in X : \dot{\phi} \in X \text{ and } \dot{\phi}(0) = \sum_{j=0}^{m} A_j \phi(-\tau_j) \}.$$

- Suppose that $A$ has $n_c$ **critical eigenvalues**/characteristic roots:

\[ \mathbb{R}(\lambda) \]

\[ \sigma_s(A) \]

\[ \sigma_u(A) \]

\[ \sigma_c(A) \]

\[ n_s = \infty \]

\[ n_c \]

\[ n_u \]
Center Manifold Theorem

- Let $S_\alpha(t) : X \to X$ be the semigroup generated by a smooth DDE
  $$\dot{x}(t) = f(x(t), x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_m), \alpha).$$
- Suppose, at $\alpha = 0$ the equilibrium $x = 0$ has critical eigenvalues. Denote by $X_0$ the (generalized) critical eigenspace of $A$ with
  \[ \dim X_0 = n_c < \infty. \]
- For each sufficiently small $|\alpha|$, there exists a smooth $n_c$-dimensional manifold $W^{c}_\alpha$ (called center manifold) that is locally invariant and normally hyperbolic for $S_\alpha(t)$. Moreover, $W^{c}_0$ is tangent at $x = 0$ to $X_0$. 

![Center Manifold Theorem Diagram](image-url)
3. Local bifurcations in one-parameter DDEs

The restriction of $S_\alpha(t)$ to $W^c_\alpha$ is locally generated by a smooth ODE

$$\dot{\xi} = g(\xi, \alpha), \; \xi \in \mathbb{R}^{nc}, \alpha \in \mathbb{R}.$$  

Fold:

- ODE on $W^c_0$: $\dot{\xi} = b_0\xi^2 + \ldots, \; \xi \in \mathbb{R}$

- Smooth normal form on $W^c_\alpha$ when $b_0 \neq 0$:

$$\dot{\xi} = \beta(\alpha) + b(\alpha)\xi^2 + \ldots, \; \beta(0) = 0, b(0) = b_0.$$  

Equilibria: $\xi_{1,2} \approx \pm \sqrt{-\beta / b}$
Andronov-Hopf:

- Normalized ODE on $W^c_0$: $\dot{z} = i\omega_0 z + c_1 z|z|^2 + \ldots$, $\omega_0 > 0, z \in \mathbb{C}$.
- Smooth normal form on $W^c_{\alpha}$ when $l_1 := \frac{1}{\omega_0} \Re(c_1) \neq 0$:

  $$\dot{z} = (\beta(\alpha) + i\omega(\alpha))z + c(\alpha)z|z|^2 + \ldots, \quad \beta(0) = 0, \omega(0) = \omega_0, c(0) = c_1.$$

Limit cycle: \( \begin{align*}
\dot{\rho} &= \rho(\beta + \Re(c)\rho^2) + \ldots, \\
\dot{\phi} &= \omega + \Im(c)\rho^2 + \ldots,
\end{align*} \)

$$\Rightarrow \rho_0 \approx \sqrt{-\frac{\beta}{\Re(c)}}$$
4. The fold critical normal form coefficient

- Let \( G(u, \alpha) := -f(u, u, u, \ldots, u, \alpha) \), so that
  \[
  J = D_u G(0, 0) = -A_0 - \sum_{j=1}^{m} A_j = \Delta(0),
  \]
  where \( A_j = D_j f(0), j = 0, 1, \ldots, m \).

- Let \( q \in \mathbb{R}^n, p \in \mathbb{R}^{n^*} \) be such that \( Jq = 0 \) and \( pJ = 0 \) with \( p\Delta'(0)q \neq 0 \).

- There is a coordinate \( \xi \in \mathbb{R} \) on \( W^c_0 \) such that the normal form coefficient
  \[
  b_0 = \frac{1}{2} p D^2_u G(0, 0)(q, q).
  \]
  This follows from the approximation of the curve \( G(u, \alpha) = 0 \) near \( (u, \alpha) = (0, 0) \in \mathbb{R}^{n+1} \), which is the finite-dimensional problem.
5. Sun-star calculus

- Consider a **nonlinear DDE**
  \[
  \dot{x}(t) = Lx_t + F(x_t), \quad x_t \in X = C([-h, 0], \mathbb{R}^n),
  \]
  where \( F : X \to \mathbb{R}^n \) is smooth and contains only nonlinear terms, while
  \[
  L\phi = \int_0^h d\zeta(\theta)\phi(-\theta), \quad \zeta \in \text{NBV}([0, h], \mathbb{R}^{n \times n}).
  \]
  Here \( \text{NBV}([0, h], \mathbb{R}^{n \times n}) \) is the space of normalized bounded-variation matrix-valued functions.

- The **linearized DDE** \( \dot{y} = Lx_t \) defines the strongly continuous semigroup \( T(t) : X \to X \) with the infinitesimal generator \( A \). Its characteristic matrix can now be written as
  \[
  \Delta(\lambda) = \lambda I_n - \int_0^h e^{-\lambda \theta} d\zeta(\theta)
  \]
Duality:

- Let $T^*(t) : X^* \to X^*$ be the **adjoint semigroup**, i.e. for $t \geq 0$

  \[ \langle T^*(t)\phi^*, \phi \rangle = \langle \phi^*, T(t)\phi \rangle, \quad \phi^* \in X^*, \phi \in X. \]

  Denote by $X^\odot$ the maximal subspace of $X^*$ on which $T^*(t)$ is strongly continuous, and define

  \[ T^\odot(t) = T^*(t)|_{X^\odot} \]

  Denote its infinitesimal generator by $A^\odot$.

- Let $A^{\odot*}$ be the generator of the adjoint semigroup $T^{\odot*}(t) : X^{\odot*} \to X^{\odot*}$. Define $X^{\odot\odot}$ as the maximal subspace of $X^{\odot*}$ on which $T^{\odot*}(t)$ is strongly continuous.

- Introduce the **embedding** $j : X \to X^{\odot*}$ by

  \[ \langle jx, x^\odot \rangle := \langle x^\odot, x \rangle, \quad \forall x \in X, \forall x^\odot \in X^\odot. \]

  In general, $j(X) \subset X^{\odot\odot}$. However, the space $X = C([-h, 0], \mathbb{R}^n)$ is **sun-reflexive**: $j(X) = X^{\odot\odot}$. 
Concrete representations:

- **Spaces:**

<table>
<thead>
<tr>
<th>space</th>
<th>representation</th>
<th>duality pairing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$\phi \in C([-h, 0], \mathbb{R}^n)$</td>
<td>$\langle \eta, \phi \rangle = \int_0^h d\eta(\theta) \phi(-\theta)$</td>
</tr>
<tr>
<td>$X^*$</td>
<td>$\eta \in NBV([0, h], \mathbb{R}^{n*})$</td>
<td></td>
</tr>
<tr>
<td>$X$</td>
<td>$\phi \in C([-h, 0], \mathbb{R}^n)$</td>
<td>$\langle (c, g), \phi \rangle = c \phi(0) + \int_0^h g(\theta) \phi(-\theta) d\theta$</td>
</tr>
<tr>
<td>$X^\circ$</td>
<td>$(c, g) \in \mathbb{R}^{n*} \times L^1([0, h], \mathbb{R}^{n*})$</td>
<td></td>
</tr>
<tr>
<td>$X^\circ^*$</td>
<td>$(\alpha, \psi) \in \mathbb{R}^n \times L^\infty([-h, 0], \mathbb{R}^n)$</td>
<td>$\langle (\alpha, \psi), (c, g) \rangle = c \alpha + \int_0^h g(\theta) \psi(-\theta) d\theta$</td>
</tr>
</tbody>
</table>

- **Embedding:**

$$j \phi = (\phi(0), \phi) \in X^\circ^*, \quad \phi \in X.$$

- **Nonlinearity** $R : X \to X^\circ^*$ is defined by

$$R(\phi) := \sum_{i=1}^n F_i(\phi)r_i^\circ^*, \quad \phi \in X,$$

where

$$r_i^\circ^* := (e_i, 0) \in X^\circ^*, \quad i = 1, \ldots, n,$$

and $e_i$ is the $i$-th standard basis vector in $\mathbb{R}^n$. 

6. The first Lyapunov coefficient for Andronov-Hopf bifurcation

- The solution $u(t) := x_t \in W_0^c \subset X$ satisfies a well-defined ODE in $X^{\odot*}$:

$$
\frac{d}{dt} j u(t) = A^{\odot*} j u(t) + R(u(t)),
$$

where $R : X \to X^{\odot*}$ can be expanded as

$$
R(u) = \frac{1}{2} B(u, u) + \frac{1}{6} C(u, u, u) + O(\|u\|^4).
$$

- The parametrization of $W_0^c$: $u = \mathcal{H}(z, \bar{z})$ with

$$
\mathcal{H}(z, \bar{z}) = z\phi + \bar{z}\phi + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} h_{j,k} z^j \bar{z}^k + O(|z|^4), \quad z \in \mathbb{C},
$$

where $A\phi = i\omega_0 \phi$, $A^* \phi^{\odot} = i\omega_0 \phi^{\odot}$, $\langle \phi^{\odot}, \phi \rangle = 1$.

- Poincaré normal form on $W_0^c$:

$$
\dot{z} = i\omega_0 z + c_1 z |z|^2 + O(|z|^4), \quad z \in \mathbb{C}.
$$
- **Homological equation**

\[ j (D_z \mathcal{H}(z, \bar{z}) \dot{z} + D_{\bar{z}} \mathcal{H}(z, \bar{z}) \dot{\bar{z}}) = A^{\odot*} j \mathcal{H}(z, \bar{z}) + R(\mathcal{H}(z, \bar{z})) \]

- **Quadratic terms**

\[
\begin{align*}
 z^2 &: -A^{\odot*} j h_{20} = B(\phi, \bar{\phi}), \\
 z\bar{z} &: (2i\omega_0 - A^{\odot*}) j h_{11} = B(\phi, \phi),
\end{align*}
\]

which are uniquely solvable and define \( h_{20} \) and \( h_{11} \).

- **Resonance cubic term**

\[
\begin{align*}
 z^2 \bar{z} &: (i\omega_0 I - A^{\odot*}) j h_{21} = C(\phi, \phi, \bar{\phi}) + B(\bar{\phi}, h_{20}) + 2B(\phi, h_{11}) - 2c_1 j \phi.
\end{align*}
\]

This system is **singular**. Pairing with \( \phi^{\odot} \) gives

\[
c_1 = \frac{1}{2} \langle \phi^{\odot}, C(\phi, \phi, \bar{\phi}) + B(\bar{\phi}, h_{20}) + 2B(\phi, h_{11}) \rangle
\]

- The **first Lyapunov coefficient**

\[ l_1 = \frac{1}{\omega_0} \Re(c_1). \]
Computational formulas:

- Eigenfunctions

\[ \phi(\theta) = e^{i\omega_0 \theta} q, \]
\[ \phi^\circ = \left( p, p \int_0^h e^{i\omega_0 (\theta - \tau)} d\zeta(\tau) \right) \]

where \( q \in \mathbb{C}^n, p \in \mathbb{C}^{n*} \) satisfy

\[ \Delta(i\omega_0)q = 0, \quad p\Delta(i\omega_0) = 0, \quad p\Delta'(i\omega_0)q = 1. \]

- Quadratic coefficients

\[ h_{20} = e^{2i\omega_0 \theta} \Delta(2i\omega_0)^{-1} D^2 F(0)(\phi, \phi) \]
\[ h_{11} = \Delta(0)^{-1} D^2 F(0)(\phi, \bar{\phi}) \]

- The normal form coefficient

\[ c_1 = \frac{1}{2} p \left[ D^2 F(0)(\bar{\phi}, e^{2i\omega_0 \theta} \Delta(2i\omega_0)^{-1} D^2 F(0)(\phi, \phi)) \right. \]
\[ + 2 D^2 F(0)(\phi, \Delta(0)^{-1} D^2 F(0)(\phi, \bar{\phi})) + D^3 F(0)(\phi, \phi, \bar{\phi}) \]

(implemented in **DDE-BIFTOOL** to compute the first Lyapunov coefficient \( l_1 \)).
Computation of derivatives: DDE at the critical parameter values:

\[
\dot{x}(t) = f(x(t), x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_m))
\]

- Recall that \( f : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n \) with

\[
X := (x^0, x^1, x^2, \ldots, x^m) \mapsto f(x^0, x^1, x^2, \ldots, x^m), \quad x^j \in \mathbb{R}^n, j = 0, 1, 2, \ldots, m.
\]

- The (multi-)linear forms:

For \( Q, P, R \in \mathbb{R}^{n(m+1)} \) with components \( q^j_k, p^j_k, r^j_k \) define

\[
D^2 f^0(Q, P) := \sum_{k_1, k_2 = 1}^{n} \sum_{j_1, j_2 = 0}^{m} \frac{\partial^2 f(0)}{\partial x_{k_1}^{j_1} \partial x_{k_2}^{j_2}} q_{k_1}^{j_1} p_{k_2}^{j_2}
\]

\[
D^3 f^0(Q, P, R) := \sum_{k_1, k_2, k_3 = 1}^{n} \sum_{j_1, j_2, j_3 = 0}^{m} \frac{\partial^3 f(0)}{\partial x_{k_1}^{j_1} \partial x_{k_2}^{j_2} \partial x_{k_3}^{j_3}} q_{k_1}^{j_1} p_{k_2}^{j_2} r_{k_3}^{j_3}
\]
Computation of derivatives:

\[ F : C([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n, \ F(\phi) = f(\phi(0), \phi(-\tau_1), \phi(-\tau_2), \ldots, \phi(-\tau_m)) \]

- 2nd Differentials:

\[
D^2 F(0)(\phi, \phi) = D^2 f^0(\Phi, \Phi),
\]
\[
D^2 F(0)(\phi, \bar{\phi}) = D^2 f^0(\Phi, \bar{\Phi}),
\]
\[
D^2 F(0)(\bar{\phi}, h_{20}) = D^2 f^0(\bar{\Phi}, H_{20}),
\]
\[
D^2 F(0)(\phi, h_{11}) = D^2 f^0(\Phi, H_{11}),
\]

where

\[
\Phi = (\phi(0), \phi(-\tau_1), \ldots, \phi(-\tau_m)),
\]
\[
H_{20} = (h_{20}(0), h_{20}(-\tau_1), \ldots, h_{20}(-\tau_m)),
\]
\[
H_{11} = (h_{11}(0), h_{11}(-\tau_1), \ldots, h_{11}(-\tau_m)).
\]

- 3rd Differential:

\[
D^3 F(0)(\phi, \phi, \bar{\phi}) = D^3 f^0(\Phi, \Phi, \bar{\Phi}).
\]