Codim 1 and 2 bifurcations of planar ODEs

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1. SOLUTIONS, ORBITS, AND PHASE PORTRAITS

General planar system:

\[
\begin{cases}
\dot{x} = P(x, y), & \text{or } \dot{X} = f(X), \quad X \in \mathbb{R}^2, \\
\dot{y} = Q(x, y)
\end{cases}
\]

where

\[
X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(X) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}.
\]

**Theorem 1** If \( f \) is smooth then for any initial point \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \) there exists a unique locally defined solution \( t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \) such that \( x(0) = x_0 \) and \( y(0) = y_0 \).
**Definition 1** Let $I$ be the maximal definition interval of a solution $t \mapsto X(t)$, $t \in I$. The oriented by the advance of time image $X(I) \subset \mathbb{R}^2$ is called the orbit.

**Vector field:** $X \mapsto f(X)$

$f(X) \neq 0$ is tangent to the orbit through $X$ \implies orbits do not cross.

**Definition 2** Phase portrait of a planar system is the collection of all its orbits in $\mathbb{R}^2$.

We draw only key orbits, which determine the topology of the phase portrait.
Types of orbits:

1. **Equilibria**: \( X(t) = X_0 \) so that \( f(X_0) = 0 \).

2. **Periodic orbits (cycles)**: \( X(t) \not\equiv X_0 \), \( X(t + T) = X(t), t \in \mathbb{R} \)
   The minimal \( T > 0 \) is called the **period** of the cycle.

3. **Connecting orbits**: \( \lim_{t \to \pm\infty} X(t) = X_\pm \) with \( f(X_\pm) = 0 \).
   If \( X_- = X_+ \) the connecting orbit is called **homoclinic**
   If \( X_- \not= X_+ \) the connecting orbit is called **heteroclinic**.

4. **All other orbits**
2. **EQUILIBRIA**  
\[ f(X) = 0 \Leftrightarrow \begin{cases} P(x, y) = 0, \\ Q(x, y) = 0. \end{cases} \]

**Jacobian matrix** of the equilibrium \( X_0 = (x_0, y_0) \):
\[
A = f_X(X_0) = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \bigg|_{x=x_0, y=y_0}
\]

**Eigenvalues** of the equilibrium \( X_0 \) are the eigenvalues of \( A \), i.e. the solutions of
\[
\lambda^2 - \sigma \lambda + \Delta = 0,
\]
where
\[
\sigma = \lambda_1 + \lambda_2 = \text{Tr} A = P_x(x_0, y_0) + Q_y(x_0, y_0), \\
\Delta = \lambda_1 \lambda_2 = \det A = P_x(x_0, y_0)Q_y(x_0, y_0) - P_y(x_0, y_0)Q_x(x_0, y_0).
\]
\[
\lambda_{1,2} = -\frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - \Delta}
\]

**Definition 3** An equilibrium \( X_0 \) is **hyperbolic** if \( \Re(\lambda) \neq 0 \).
Phase portraits of generic planar systems $\dot{Y} = AY$

<table>
<thead>
<tr>
<th>$(n_u, n_s)$</th>
<th>Eigenvalues</th>
<th>Phase portrait</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2)</td>
<td></td>
<td><img src="image" alt="Node" /></td>
<td>node</td>
</tr>
<tr>
<td>(1, 1)</td>
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<td><img src="image" alt="Focus" /></td>
<td>focus</td>
</tr>
<tr>
<td>(2, 0)</td>
<td></td>
<td><img src="image" alt="Focus" /></td>
<td>focus</td>
</tr>
</tbody>
</table>
Definition 4 Two systems are called topologically equivalent if their phase portraits are homeomorphic, i.e. there is a continuous invertible transformation that maps orbits of one system onto orbits of the other, preserving their orientation.

Theorem 2 (Grobman-Hartman) Consider a smooth nonlinear system

\[ \dot{X} = AX + F(X), \quad F = O(\|X\|^2) \equiv O(2), \]

and its linearization

\[ \dot{Y} = AY. \]

If \( \Re(\lambda) \neq 0 \) for all eigenvalues of \( A \), then these systems are locally topologically equivalent near the origin.

Warning: A stable/unstable node is locally topologically equivalent to a stable/unstable focus.
Trivial topological equivalences

1. Orbital equivalence:

\[ \dot{X} = f(X) \sim \dot{Y} = g(Y)f(Y) \]

where \( g : \mathbb{R}^2 \to \mathbb{R} \) is a smooth positive function; \( Y = h(X) = X \) preserves orbits.

2. Smooth equivalence:

\[ \dot{X} = f(X) \sim \dot{Y} = [h_Y(Y)]^{-1}f(h(Y)), \]

where \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) is a smooth diffeomorphism; the substitution \( X = h(Y) \) transforms solutions onto solutions:

\[ \dot{X} = h_Y(Y)\dot{Y} = f(h(Y)) = f(X). \]

PERIODIC ORBITS AND LIMIT CYCLES

Poincaré map:

\[ \xi \mapsto P(\xi) = \mu \xi + O(2), \]

where the multiplier

\[ \mu = \exp \left( \int_0^T (\text{div } f) (X^0(t)) \, dt \right) > 0 \]

**Definition 5** A cycle of the planar system is **hyperbolic** if \( \mu \neq 1 \).

The cycle is stable if \( \mu < 1 \) and is unstable if \( \mu > 1 \).
HOMOCLINIC ORBITS

Homoclinic orbits to saddles:

\[
\begin{align*}
\sigma &< 0 & \text{small} & \Gamma_0 \\
\sigma &> 0 & \text{big} & \Gamma_0
\end{align*}
\]

**Definition 6** *The real number \( \sigma = \lambda_1 + \lambda_2 = (\text{div } f)(X_0) \) is called the saddle quantity of \( X_0 \).*
Near the saddle, any planar system is $C^1$-equivalent to its linearization.

**Singular map:**

\[
\begin{align*}
\dot{x} &= \lambda_1 x \\
\dot{y} &= \lambda_2 y \\
\end{align*}
\]

\[
\xi = \Delta(\eta) = \eta^{-\frac{\lambda_1}{\lambda_2}}
\]

**Regular map:**

\[
\tilde{\eta} = Q(\xi) = A \xi + O(2), \quad A > 0.
\]

**Poincaré map:**

\[
\eta \mapsto \tilde{\eta} = Q(\Delta(\eta)) = A \eta^{-\frac{\lambda_1}{\lambda_2}} + \ldots
\]

The homoclinic orbit is stable if $\sigma < 0$ and is unstable if $\sigma > 0$. 
If \( \sigma = \lambda_1 + \lambda_2 = 0 \), then

\[
\text{if } \int_{-\infty}^{\infty} (\text{div } f)(X^0(t)) \, dt < 0 \text{ the homoclinic orbit is stable;}
\]

\[
\text{if } \int_{-\infty}^{\infty} (\text{div } f)(X^0(t)) \, dt > 0 \text{ the homoclinic orbit is unstable.}
\]

**Homoclinic orbits to saddle-nodes:**
3. BIFURCATIONS AND THEIR CLASSIFICATION

Consider a smooth 2D system depending on one parameter

\[ \dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}. \]

**Definition 7** A point \( \alpha_0 \) is called a **bifurcation point** if in any neighborhood of \( \alpha_0 \) there is a point \( \alpha \) for which

\[ \dot{X} = f(X, \alpha) \not\sim \dot{X} = f(X, \alpha_0). \]

The appearance of a topologically non-equivalent system is called a **bifurcation**.

Since the number of equilibria, the number of periodic orbits, and their stability, as well as the presence of connecting orbits, are topological invariants, a bifurcation of the 2D-system means a change of (some of) these properties.
**Definition 8** A **codimension** of a bifurcation is the number of conditions on which the bifurcating phase object has to satisfy.

**Classification of codimension-one bifurcations:**

1. **Local (near equilibria)**
   - *saddle-node (fold)*
   - *(Andronov–) Hopf*

2. **Local of cycles (near periodic orbits)**
   - *(cyclic) fold*

3. **Global**
   - **Bifurcations of homo- and heteroclinic orbits**
     - *saddle homoclinic*
     - *saddle–node homoclinic*
     - *heteroclinic*

Only codim 1 bifurcations occur in generic one-parameter systems.
4. LOCAL CODIM 1 BIFURCATIONS

- If $X_0$ is a hyperbolic equilibrium of $\dot{X} = f(X, \alpha_0)$, then it remains hyperbolic for all $\alpha$ sufficiently close to $\alpha_0$ (but can slightly shift).

- A local bifurcation can happen only to a non-hyperbolic equilibrium with $\Re(\lambda) = 0$.

- Codimension-1 critical cases:
  1. Fold (saddle-node): $\lambda_1 = 0$
  2. Andronov-Hopf (weak focus): $\lambda_{1,2} = \pm i\omega$
**Fold bifurcation:** $\lambda_1 = 0$, $\lambda_2 \neq 0$

By a linear diffeomorphism, $\dot{X} = f(X,0)$ can be transformed into
$$
\begin{align*}
\dot{x} &= ax^2 + bxy + cy^2 + O(3), \\
\dot{y} &= \lambda_2 y + O(2).
\end{align*}
$$

If $a \neq 0$ then $\dot{X} = f(X)$ is locally topologically equivalent near the origin to
$$
\begin{align*}
\dot{x} &= ax^2, \\
\dot{y} &= \lambda_2 y.
\end{align*}
$$

**Saddle-node** ($a > 0$):

\[
\begin{array}{l}
\lambda_2 < 0 \\
\lambda_2 > 0
\end{array}
\]
Theorem 3 (Fold normal form) If \( a \neq 0 \) and \( \lambda_2 \neq 0 \), then \( \dot{X} = f(X, \alpha) \) is locally topologically equivalent near the saddle-node to

\[
\begin{align*}
\dot{x} &= \beta(\alpha) + ax^2, \\
\dot{y} &= \lambda_2 y,
\end{align*}
\]

where \( \beta(0) = 0 \).

Two equilibria \( O_{1,2} = \left( \mp \sqrt{-\frac{\beta}{a}}, 0 \right) \) collide and disappear in the 1D center manifold \( W_c = \{ y = 0 \} \), provided \( \beta'(0) \neq 0 \).
**Andronov-Hopf bifurcation:** $\lambda_{1,2} = \pm i\omega, \ \omega > 0$

By a linear diffeomorphism, $\dot{X} = f(X,0)$ can be transformed into

$$\begin{cases}
\dot{x} = -\omega y + R(x,y), & R = O(2), \\
\dot{y} = \omega x + S(x,y), & S = O(2).
\end{cases}$$

Introduce $z = x + iy \in \mathbb{C}$. Then this system becomes

$$\dot{z} = i\omega z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = R\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iS\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right).$$

Write its Taylor expansion in $z, \bar{z}$:

$$g(z,\bar{z}) = \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \ldots$$

**Definition 9** *The first Lyapunov coefficient* is

$$l_1 = \frac{1}{2\omega^2} \Re(ig_{20}g_{11} + \omega g_{21}).$$
If $l_1 \neq 0$ then $\dot{X} = f(X)$ is locally topologically equivalent near the origin to

$$\begin{cases}
\dot{\rho} = l_1 \rho^3, \\
\dot{\varphi} = 1,
\end{cases}$$

where $(\rho, \varphi)$ are polar coordinates: $z = \rho e^{i\varphi}$.

**Weak focus:**

![Diagram showing stable and unstable foci with $l_1 < 0$ and $l_1 > 0$]
Theorem 4 (Andronov-Hopf normal form) If $l_1 \neq 0$ and $\omega > 0$, then $\dot{X} = f(X,\alpha)$ is locally topologically equivalent near the weak focus to

$$\begin{align*}
\dot{\rho} &= \rho(\beta(\alpha) + l_1 \rho^2), \\
\dot{\phi} &= 1.
\end{align*}$$

where $\beta(0) = 0$.

A limit cycle $\rho_0 = \sqrt{-\beta / l_1} > 0$ appears while the focus changes stability.

The direction of the cycle bifurcation is determined by the first Lyapunov coefficient $l_1$ of the weak focus:

- **supercritical** (soft, non-catastrophic) Andronov-Hopf bifurcation ($l_1 < 0$);
- **subcritical** (hard, catastrophic) Andronov-Hopf bifurcation ($l_1 > 0$).
Supercritical Andronov-Hopf bifurcation: $l_1 < 0$

The stable equilibrium is replaced by small-amplitude oscillations within an attracting domain.
Subcritical Andronov-Hopf bifurcation: $l_1 > 0$

The domain of attraction of the stable focus shrinks, while it becomes unstable.
Example: \[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x + \alpha y + x^2 + xy + y^2.
\end{align*}
\]

At \(\alpha = 0\) the equilibrium \(x = y = 0\) of the reversed system
\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x - x^2 - xy - y^2,
\end{align*}
\]
has eigenvalues \(\lambda_{1,2} = \pm i\) \((\omega = 1)\).

Introduce \(z = x + iy\), then \(x^2 + y^2 = |z|^2 = z\bar{z}\) and
\[
\dot{z} = \dot{x} + iy = -y + ix - ix^2 - ixy - iy^2 = iz - iz\bar{z} - \frac{1}{4}(z^2 - \bar{z}^2) = iz - \frac{1}{4}z^2 - iz\bar{z} + \frac{1}{4}\bar{z}^2
\]
so that \(\omega = 1\), \(g_{20} = -\frac{1}{2}\), \(g_{11} = -i\), \(g_{02} = \frac{1}{2}\), \(g_{21} = 0\).

\[
\bar{l}_1 = \frac{1}{2\omega^2} \Re(ig_{20}g_{11} + \omega g_{21}) = \frac{1}{2} \left( i \frac{1}{2} i + 1 \cdot 0 \right) = -\frac{1}{4}.
\]

For the original system, \(l_1 = \frac{1}{4} > 0 \Rightarrow \) subcritical Hopf bifurcation (an unstable cycle exists for small \(\alpha < 0\) but disappears for \(\alpha > 0\))
Practical computation of $a$ and $l_1$ in $\mathbb{R}^2$ ($n = 2$)

Suppose $X_0 = 0$, $\alpha_0 = 0$ and write the Taylor expansion in the original coordinates:

$$f(X, 0) = AX + \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X) + O(4)$$

where

$$(AX)_i = \sum_{j=1}^{n} \frac{\partial f_i(U, 0)}{\partial U_j} \bigg|_{U=0} X_j,$$

$$B_i(X, Y) = \sum_{j,k=1}^{n} \frac{\partial^2 f_i(U, 0)}{\partial U_j \partial U_k} \bigg|_{U=0} X_j Y_k,$$

$$C_i(X, Y, Z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 f_i(U, 0)}{\partial U_j \partial U_k \partial U_l} \bigg|_{U=0} X_j Y_k Z_l,$$

for $i = 1, \ldots, n.$
Theorem 5  The fold normal form coefficient can be computed as

\[ a = \frac{1}{2} \langle p, B(q, q) \rangle \]

where \( p, q \in \mathbb{R}^2 \) satisfy

\[ Aq = A^T p = 0 \]

and \( p^T q \equiv \langle p, q \rangle = 1 \).

Theorem 6  The first Lyapunov coefficient can be computed in 2D as

\[ l_1 = \frac{1}{2\omega^2} \Re \left[ i \langle p, B(q, q) \rangle \langle p, B(q, \bar{q}) \rangle + \omega \langle p, C(q, q, \bar{q}) \rangle \right] \]

where \( p, q \in \mathbb{C}^2 \) satisfy

\[ Aq = i\omega q, \quad A^T p = -i\omega p \]

and \( \bar{p}^T q \equiv \langle p, q \rangle = 1 \).
Example: Hopf bifurcation in a prey-predator system

Consider the following system

\[
\begin{cases}
\dot{x}_1 = rx_1(1 - x_1) - \frac{cx_1x_2}{\alpha + x_1} \\
\dot{x}_2 = -dx_2 + \frac{cx_1x_2}{\alpha + x_1}
\end{cases}
\sim
\begin{cases}
\dot{x}_1 = rx_1(\alpha + x_1)(1 - x_1) - cx_1x_2 \\
\dot{x}_2 = -\alpha dx_2 + (c - d)x_1x_2
\end{cases}
\]

At \( \alpha_0 = \frac{c-d}{c+d} \) the last system has the equilibrium \( (x_1^{(0)}, x_2^{(0)}) = \left( \frac{d}{c+d}, \frac{rc}{(c+d)^2} \right) \) with eigenvalues \( \lambda_{1,2} = \pm i\omega \), where \( \omega^2 = \frac{rc^2d(c-d)}{(c+d)^3} > 0 \).

Translate the origin of the coordinates to this equilibrium by

\[
\begin{cases}
x_1 = x_1^{(0)} + X_1, \\
x_2 = x_2^{(0)} + X_2.
\end{cases}
\]
This transforms the system into
\[
\begin{align*}
\dot{X}_1 &= -\frac{cd}{c+d}X_2 - \frac{rd}{c+d}X_1^2 - cX_1X_2 - rX_1^3, \\
\dot{X}_2 &= \frac{rc(c-d)}{(c+d)^2}X_1 + (c-d)X_1X_2,
\end{align*}
\]
that can be represented as
\[
\dot{\mathbf{x}} = A\mathbf{x} + \frac{1}{2}B(\mathbf{x},\mathbf{x}) + \frac{1}{6}C(\mathbf{x},\mathbf{x},\mathbf{x}),
\]
where
\[
A = \begin{pmatrix}
0 & -\frac{cd}{c+d} \\
\frac{\omega^2(c+d)}{cd} & 0
\end{pmatrix},
B(\mathbf{x},\mathbf{y}) = \begin{pmatrix}
-\frac{2rd}{c+d}X_1Y_1 - c(X_1Y_2 + X_2Y_1) \\
(c-d)(X_1Y_2 + X_2Y_1)
\end{pmatrix}
\]
and
\[
C(\mathbf{x},\mathbf{y},\mathbf{z}) = \begin{pmatrix}
-6rX_1Y_1Z_1 \\
0
\end{pmatrix}.
\]
The complex vectors
\[ q = \begin{pmatrix} cd \\ -i\omega(c + d) \end{pmatrix}, \quad p = \frac{1}{2\omega cd(c + d)} \begin{pmatrix} \omega(c + d) \\ -icd \end{pmatrix}. \]
satisfy \( Aq = i\omega q, \ A^T p = -i\omega p \) and \( \langle p, q \rangle = 1. \)

Then
\[ g_{20} = \langle p, B(q, q) \rangle = \frac{cd(c^2 - d^2 - rd) + i\omega c(c + d)^2}{(c + d)}, \]
\[ g_{11} = \langle p, B(q, \bar{q}) \rangle = -\frac{r cd^2}{(c + d)}, \quad g_{21} = \langle p, C(q, q, \bar{q}) \rangle = -3rc^2d^2, \]
and the first Lyapunov coefficient
\[ l_1(\alpha_0) = \frac{1}{2\omega^2} \text{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rc^2d^2}{\omega} < 0. \]
Therefore, a **stable cycle** bifurcates from the equilibrium via the super-critical Hopf bifurcation for $\alpha < \alpha_0$.

One can prove that the cycle is **unique**.
5. CODIM 1 CYCLIC FOLD BIFURCATION

Parameter-dependent Poincaré map:

\[ \xi \mapsto \tilde{\xi} = P(\xi, \alpha), \]

where \( P(\xi, 0) = \xi + O(2) \quad (\mu = 1) \)

**Lemma 1** If

\[ p_2(0) = \frac{1}{2} P_{\xi\xi}(0, 0) \neq 0, \]

then there exists a smooth function \( \delta = \delta(\alpha) \) such that the substitution \( x = \xi + \delta(\alpha) \) reduces the map

\[ \xi \mapsto P(\xi, \alpha) = p_0(\alpha) + [1 + g(\alpha)]\xi + p_2(\alpha)\xi^2 + O(3), \]

where \( g(0) = 0, p_0(0) = P(0, 0) = 0, \) to the form

\[ x \mapsto \tilde{x} = \beta(\alpha) + x + b(\alpha)x^2 + O(3) \]

with \( \beta(0) = 0 \) and \( b(0) = p_2(0) \neq 0. \)
Cyclic fold: \( x \mapsto \beta + x + bx^2, \ b > 0 \)

Two hyperbolic cycles (unstable \( C_1 \) and stable \( C_2 \)) collide forming a non-hyperbolic cycle \( C_0 \), and disappear.
6. CODIM 1 BIFURCATIONS OF CONNECTING ORBITS

- Saddle homoclinic bifurcation

Singular map: \( \eta \mapsto \xi = \eta^{-\frac{\lambda_1}{\lambda_2}}. \)

Regular map:
\[
\xi \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\xi + O(2), \quad A(0) > 0.
\]

Poincaré map:
\[
\eta \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\eta^{-\frac{\lambda_1}{\lambda_2}} + \ldots
\]
Saddle homoclinic bifurcation: $\sigma < 0$

A stable cycle $C_\beta$ bifurcates from $\Gamma_0$ while the separatrices exchange.
Saddle homoclinic bifurcation: $\sigma > 0$

An unstable cycle $C_{\beta}$ bifurcates from $\Gamma_0$ while the separatrices exchange.
• Homoclinic saddle-node bifurcation:

\[ \Gamma_0 \]

\[ \beta < 0 \]
\[ \beta = 0 \]
\[ \beta > 0 \]

• Heteroclinic saddle bifurcation:

\[ C_\beta \]

\[ \beta < 0 \]
\[ \beta = 0 \]
\[ \beta < 0 \]
Example: Allee effect in a prey-predator system

\[
\begin{align*}
\dot{x} &= x(x - l)(1 - x) - xy, \\
\dot{y} &= -\gamma y(m - x).
\end{align*}
\]
7. LOCAL CODIM 2 BIFURCATIONS

Consider a smooth 2D system depending on two parameters

\[ \dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^2. \]

Curves of codim 1 bifurcations:

- **Fold**:
  \[ \begin{align*}
  & f(X, \alpha) = 0, \\
  & \det f_X(X, \alpha) = 0.
  \end{align*} \]

- **Hopf**:
  \[ \begin{align*}
  & f(X, \alpha) = 0, \\
  & \text{Tr} \ f_X(X, \alpha) = 0.
  \end{align*} \]

In both cases, we have 3=2+1 equations in \( \mathbb{R}^4 \).

When we cross \( B = \pi \Gamma \) in the \( \alpha \)-plane, the corresponding codim 1 bifurcation occurs.

One has to check that \( \lambda_{1,2} = \pm i\omega \) along the Hopf curve.
Local codim 2 cases in the plane:

Fold: $\lambda_1 = 0$
\[
\begin{align*}
\dot{x} &= ax^2 + O(3) \\
\dot{y} &= \lambda_2 y + O(2)
\end{align*}
\]

Hopf: $\lambda_{1,2} = \pm i\omega$
\[
\begin{align*}
\dot{\rho} &= \omega l_1 \rho^3 + O(4) \\
\dot{\phi} &= \omega + O(1)
\end{align*}
\]

To meet each case, we need to “tune” two parameters while following $\Gamma$ (or $B$) $\Rightarrow$ codim 2.

1. $\lambda_1 = 0, \ a = 0$
2. $\lambda_1 = 0, \ \lambda_2 = 0$
3. $\lambda_{1,2} = \pm i\omega, \ l_1 = 0$
Cusp bifurcation: $\lambda_1 = 0, \ a = 0$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to

$$
\begin{cases}
\dot{x} = p_{11}xy + \frac{1}{2}p_{02}y^2 + \frac{1}{6}p_{30}x^3 + \ldots, \\
\dot{y} = \lambda_2 y + \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + O(3).
\end{cases}
$$

It has an invariant 1D center manifold $W^c = \{(x, y) : y = W(x)\}$:

$$
y = W(x) = \frac{1}{2}w_2x^2 + O(3)
$$

where $w_2 = -\frac{q_{20}}{\lambda_2}$.

Thus, the restriction of $\dot{X} = f(X, 0)$ to $W^c$ is

$$
\dot{x} = cx^3 + O(4), \text{ where } c = \frac{1}{6} \left( p_{30} - \frac{3}{\lambda_2}q_{20}p_{11} \right).
$$
Theorem 7 (Cusp normal form) \[ \text{If } c \neq 0, \text{ then } \dot{X} = f(X, \alpha) \text{ is locally topologically equivalent near the cusp bifurcation to} \]
\[
\begin{cases}
\dot{x} = \beta_1(\alpha) + \beta_2(\alpha)x + sx^3, \\
\dot{y} = \lambda_2 y,
\end{cases}
\]

where \( \beta_1(0) = \beta_2(0) = 0 \) and \( s = \text{sign}(c) = \pm 1 \).

Fold curve(s) \( 4\beta_2^3 + 27s\beta_1^2 = 0 \)
Cusp bifurcation diagram \((c < 0, \lambda_2 < 0)\)

Three equilibria exist inside the wedge, pairwise colliding at its borders \(T_{1,2}\) and leaving one equilibrium outside.
**Bogdanov-Takens bifurcation:** \( \lambda_1 = \lambda_2 = 0 \)

The critical system \( \dot{X} = f(X, 0) \) can be transformed by a linear diffeomorphism to

\[
\begin{align*}
\dot{x} &= y + \frac{1}{2}p_{20}x^2 + p_{11}xy + \frac{1}{2}p_{02}y^2 + O(3) \equiv P(x, y), \\
\dot{y} &= \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + \frac{1}{6}q_{03}x^2 + O(3).
\end{align*}
\]

By a nonlinear local diffeomorphism (change of variables)

\[
\begin{align*}
\xi &= x, \\
\eta &= P(x, y),
\end{align*}
\]

this system can be reduced near the origin to

\[
\begin{align*}
\dot{\xi} &= \eta, \\
\dot{\eta} &= a\xi^2 + b\xi\eta + \ldots,
\end{align*}
\]

where

\[
a = \frac{1}{2}q_{20}, \quad b = p_{20} + q_{11}.
\]
Theorem 8 (Bogdanov-Takens normal form) \( \text{If } ab \neq 0, \text{ then} \)

\[
\dot{X} = f(X, \alpha)
\]

is locally topologically equivalent near the BT-bifurcation to

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \beta_1(\alpha) + \beta_2(\alpha)x + x^2 + sxy,
\end{align*}
\]

where \( \beta_1(0) = \beta_2(0) = 0 \) and \( s = \text{sign}(ab) = \pm 1 \).

Bifurcation curves \((ab < 0)\):

- **fold** \( T : \beta_1 = \frac{1}{4} \beta_2^2 \)

- **Andronov-Hopf** \( H : \beta_1 = 0, \beta_2 < 0 \)

- **saddle homoclinic** \( P : \beta_1 = -\frac{6}{25} \beta_2^2 + O(3), \beta_2 < 0 \) (global bifurcation)
BT bifurcation diagram \((ab < 0)\)

A unique limit cycle appears at Andronov-Hopf bifurcation curve \(H\) and disappears via the saddle homoclinic orbit at the curve \(P\).
Bautin ("generalized Hopf") bifurcation: $\lambda_{1,2} = \pm i \omega, \ l_1 = 0$

The critical system $\dot{X} = f(X,0)$ can be transformed by a linear diffeomorphism to the complex form

$$\dot{z} = i \omega z + \sum_{2 \leq j + k \leq 5} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(6),$$

which is locally smoothly equivalent to the Poincaré normal form

$$\dot{w} = i \omega w + c_1 w |w|^2 + c_2 w |w|^4 + O(6),$$

where the Lyapunov coefficients

$$l_j = \frac{1}{\omega} \Re(c_j)$$

satisfy

$$2l_1 = \frac{1}{\omega} \left( \Re(g_{21}) - \frac{1}{\omega} \Im(g_{20} g_{11}) \right) \Rightarrow l_1 = \frac{1}{2\omega^2} \Re(ig_{20}g_{11} + \omega g_{21})$$
If $l_1 = 0$ then

\[
12l_2(0) = \frac{1}{\omega} \Re(g_{32})
\]
\[
+ \frac{1}{\omega^2} \Im \left[ g_{20}g_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12} \right]
\]
\[
+ \frac{1}{\omega^3} \Re \left[ g_{20}(\bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30}\right) + \frac{1}{3}\bar{g}_{02}g_{03})
\right.
\]
\[
+ g_{11}(\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12}\right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30})
\]
\[
\left. + 3\Im(g_{20}g_{11}) \Im(g_{21}) \right]
\]
\[
+ \frac{1}{\omega^4} \left\{ \Im \left[ g_{11}\bar{g}_{02}\left(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\right) \right]
\right.
\]
\[
+ \Im(g_{20}g_{11}) \left[ 3\Re(g_{20}g_{11}) - 2|g_{02}|^2 \right] \}
Theorem 9 (Normal form for Bautin bifurcation) If $l_2 \neq 0$ and $\omega \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near Bautin bifurcation to the normal form in the polar coordinates:

$$\begin{cases}
\dot{\rho} = \rho(\beta_1(\alpha) + \beta_2(\alpha)\rho^2 + s\rho^4), \\
\dot{\phi} = 1,
\end{cases}$$

where $\beta_1(0) = \beta_2(0) = 0$ and $s = \text{sign}(l_2) = \pm 1$.

Bifurcation curves $(l_2 < 0)$:

- **superctitical Andronov-Hopf** $H^- : \beta_1 = 0, \beta_2 < 0$

- **subctitical Andronov-Hopf** $H^+ : \beta_1 = 0, \beta_2 > 0$

- **cyclic fold** $T_c : \beta_1 = \frac{1}{4} \beta_2^2, \beta_2 > 0$ (global bifurcation)
In the wedge between $H^+$ and $T_c$ there exist two limit cycles born via different Andronov-Hopf bifurcations, which merge and disappear at the cyclic fold curve $T_c$. 
Example: Bazykin’s prey-predator model

\[
\begin{align*}
\dot{x}_1 &= x_1 - \frac{x_1 x_2}{1 + \alpha x_1} - \varepsilon x_1^2, \\
\dot{x}_2 &= -\gamma x_2 + \frac{x_1 x_2}{1 + \alpha x_1} - \delta x_2^2.
\end{align*}
\]
Generic phase portraits: