Local Bifurcations in Neural Field Equations

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Joint work with

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On local bifurcations in neural field models with transmission delays
J. Math. Biology: **66** (4-5) 837–887, 2013.

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Neural Field Equations

Neural activity dynamics in an open simply-connected domain $\Omega \subset \mathbb{R}^n$ is modeled by

$$\frac{\partial V}{\partial t}(t,\mathbf{r}) = -\alpha V(t,\mathbf{r}) + \int_\Omega J(\mathbf{r},\mathbf{r}')S(V(t-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}'))\,d\mathbf{r}' \qquad \text{(NFE)}$$

[Wilson & Cowan 1972; Amari 1977]

- (H_J) The **connectivity kernel** $J \in C(\overline{\Omega} \times \overline{\Omega})$.
- (H_S) the **synaptic activation function** $S \in C^\infty(\mathbb{R})$ and its kth derivative is bounded for every $k \in \mathbb{N}_0$.
- (H_τ) The **transmission delay function** $\tau \in C(\overline{\Omega} \times \overline{\Omega})$

$$0 < h := \sup\{\tau(\mathbf{r},\mathbf{r}') : \mathbf{r},\mathbf{r}' \in \Omega\} < \infty$$

Explicit 1D example

$\overline{\Omega} = [-1, 1]$ and

$$\tau(x, x') = \tau(|x - x'|) = \tau_0 + |x - x'| \qquad \forall\, x, x' \in \overline{\Omega}$$

$$S(V) = \frac{1}{1 + e^{-rV}} - \frac{1}{2} \qquad \forall\, V \in \mathbb{R}$$

For the connectivity kernel we take a linear combination of $N \geq 1$ exponentials,

$$J(x, x') = J(|x - x'|) = \sum_{i=1}^{N} c_i e^{-\mu_i |x - x'|} \qquad \forall\, x, x' \in \overline{\Omega}$$

where $c_i \in \mathbb{C}$ with $c_i \neq 0$ and $\mu_i \in \mathbb{C}$ with $\mu_i \neq \mu_j$ for $i \neq j$

Simple discretization

Introduce the uniform mesh

$$-1 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1, \quad x_{i+1} - x_i = \Delta = \frac{2}{m}.$$

Approximating each integral in

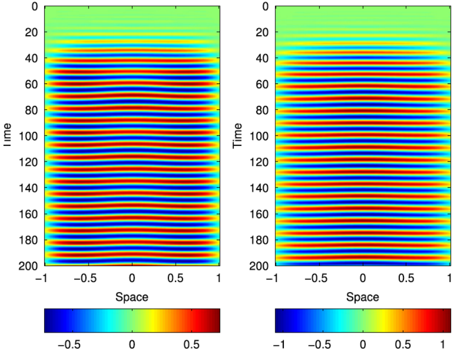
$$\frac{\partial V}{\partial t}(t,x) = -\alpha V(t,x) + \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} J(x,x') S(V(t-\tau(x,x'),x'))\, dx'$$

with the two-point trapezoid rule, one obtains for $V_j(t) = V(x_j, t)$

$$\frac{dV_j(t)}{dt} = -\alpha V_j(t) + \frac{2}{m} \sum_{i=1}^{m} w_j J(\Delta|i-j|) S(V(t-\tau_0-\Delta|i-j|)),$$

where $j = 0, 1, \ldots, m$ and $w_j = \frac{1}{2}$ for $j \in \{0, m\}$ and $w_j = 1$ for $j \notin \{0, m\}$.

Simulation with MATLAB dde23



G. Faye and O. Faugeras
Some theoretical and numerical results for delayed neural field equations
Physica D: **239** (9) 561–578, 2010.

Question

Can oscillations in NFE be explained by a Hopf bifurcation on an invariant manifold of some dynamical system in an appropriate function space X ?

- Existence of solutions
- Principle of linearized (in)stability
- Invariant (center) manifolds
- Local bifurcations and their normal forms

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Local bifurcations of codim 1

- Consider a smooth autonomous ODE system

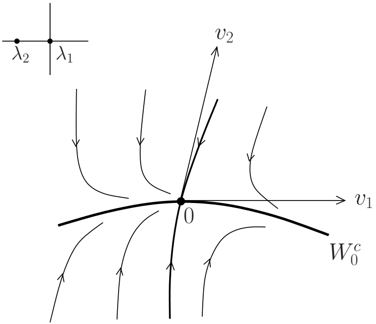
$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R} \tag{ODE}$$

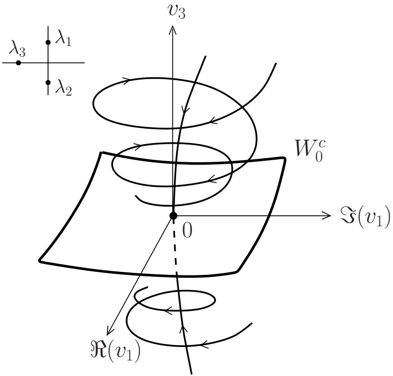
- Let $u_0 = 0$ be an equilibrium at $\alpha = 0$ with n_c critical eigenvalues.
- Simplest non-hyperbolic cases:

  - **Fold (limit point, LP)**: $\lambda_1 = 0$;
  - **Andronov-Hopf (H)**: $\lambda_{1,2} = \pm i\omega_0, \ \omega_0 > 0$.

Center manifold reduction for ODEs

There exists a local invariant **center manifold** W^c_α of dimension n_c, such that W^c_0 is tangent to the critical eigenspace of $A = D_u f(0,0)$.





Technical tools for ODEs

The standard proof of the existence of W_0^c for

$$\dot{u} = Au + R(u), \quad R(u) \in O(\|u\|^2)$$

is based on

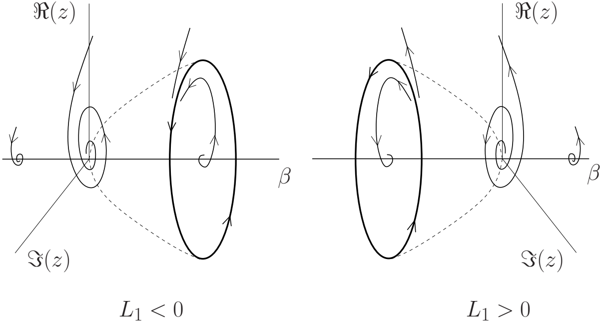
- finite-dimensionality of the phase space $\mathbb{R}^n$
- smoothness of R
- variation-of-constants formula:

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} R(u(s))\, ds$$

- $T(t) = e^{At}$ forming a group
- A being the generator of $T = \{T(t)\}_{t \in \mathbb{R}}$

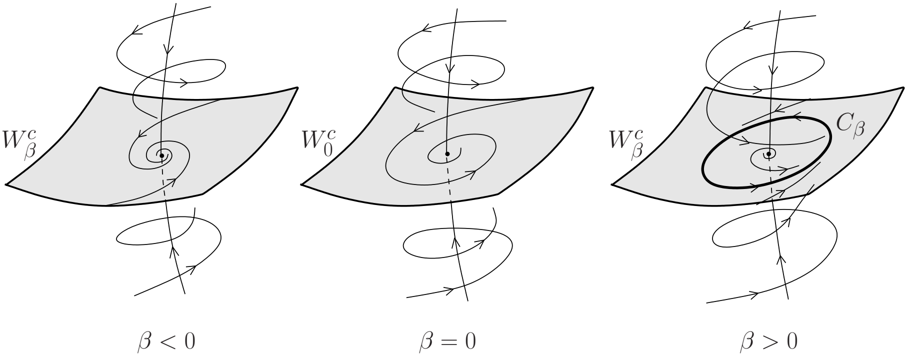
Hopf normal form on W^c_α

- $\dot z = (\beta(\alpha) + i\omega(\alpha))z + c_1(\alpha)z|z|^2 + O(|z|^4),\ \beta(0) = 0,\ \omega(0) = \omega_0 > 0.$
- **First Lyapunov coefficient**: $L_1 = \dfrac{1}{\omega_0}\Re(c_1(0)) \neq 0$



- Approximate cycle: $\begin{cases} \dot\rho = \rho(\beta + \Re(c_1)\rho^2) \\ \dot\varphi = \omega + \Im(c_1)\rho^2 \end{cases} \Rightarrow \rho_0 = \sqrt{-\dfrac{\beta}{\Re(c_1)}}$

Hopf bifurcation in $\mathbb{R}^n$



$$(n = 3, n_s = 1, n_u = 0, n_c = 2, L_1 < 0)$$

Computation of $c_1(0)$

- $Aq = i\omega_0 q, A^{\mathrm{T}} p = -i\omega_0 p, \langle q,q\rangle = \langle p,q\rangle = 1$, where $\langle p,q\rangle = \bar{p}^{\mathrm{T}} q$.
- Let

$$F(u) = Au + \frac{1}{2}B(u,u) + \frac{1}{3!}C(u,u,u) + O(\|u\|^4)$$

- Locally represent W_0^c as

$$u = H(z,\overline{z}) = zq + \overline{z}\,\overline{q} + \sum_{2\le j+k\le 3} \frac{1}{j!k!} h_{jk} z^j \overline{z}^k + O(|z|^4)$$

Assume the restriction of $\dot{u} = F(u)$ to W_0^c to be in the normal form

$$\dot{z} = G(z,\overline{z}) = i\omega_0 z + c_1(0) z|z|^2 + O(|z|^4)$$

- The invariance of W_0^c implies the **homological equation**

$$D_z H(z,\overline{z}) G(z,\overline{z}) + D_{\overline{z}} H(z,\overline{z}) \overline{G}(z,\overline{z}) = F(H(z,\overline{z}))$$

- Quadratic z^2- and $|z|^2$-terms give nonsingular linear systems

$$\begin{aligned}(2i\omega_0 I_n - A)h_{20} &= B(q,q)\\ -Ah_{11} &= B(q,\overline{q})\end{aligned}$$

- Cubic $z^2\overline{z}$-terms give the singular system

$$(i\omega_0 I_n - A)h_{21} = C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - 2c_1(0)q$$

The **Fredholm solvability** of this system implies

$$c_1(0) = \frac{1}{2}\langle p, C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11})\rangle$$

- The **first Lyapunov coefficient**

$$L_1 = \frac{1}{\omega_0}\Re(c_1(0))$$

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NFE as Abstract Delay Differential Equation

Let $Y = C(\overline{\Omega})$ and $X = C([-h,0];Y)$. Define $G : X \to Y$ by

$$G(\phi)(\mathbf{r}) = \int_{\Omega} J(\mathbf{r},\mathbf{r}')S(\phi(-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}'))\,d\mathbf{r}' \quad \forall\,\phi \in X, \forall\,\mathbf{r} \in \Omega$$

and introduce

$$x(t)(\mathbf{r}) = V(t,\mathbf{r})$$
$$x_t(\theta)(\mathbf{r}) = V(t+\theta,\mathbf{r}), \quad -h \le \theta \le 0 \ \text{(history at time t)}$$

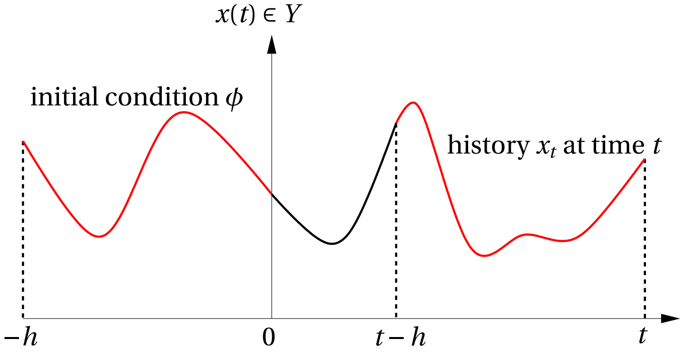
Then the NFE equation can be written as

$$\begin{cases} \dot{x}(t) = F(x_t) & t \ge 0 \\ x(t) = \phi(t) & t \in [-h,0] \end{cases} \tag{ADDE}$$

where

$$F(\phi) = -\alpha\phi(0) + G(\phi) \qquad \forall\,\phi \in X$$

ADDE as a Dynamical System



Strategy

- Embed X in a larger state space. This space is called $X^{\odot *}$, pronounce X-sun-star.
- There is a canonical way to obtain this space.
- The canonical embedding is called $j : X \mapsto X^{\odot *}$.
- On this larger subspace the translation semigroup is also defined: $T_0^{\odot *}$
- The variation-of-constants formula holds on this larger subspace.
- But if we start in X, we stay there!

O. Diekmann, S.A. van Gils, S. Verduyn Lunel, and H.-O. Walther
Delay Equations: Functional, complex, and nonlinear analysis
Applied Mathematical Sciences **110** , Springer-Verlag, 1995

The shift semigroup T_0 on X

Define a strongly continuous semigroup on X by

$$(T_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & -h \leq \theta \leq -t \\ \phi(0) & -t < \theta \leq 0 \end{cases} \qquad \forall\, \phi \in X,\ t \geq 0$$

This semigroup solves the trivial (ADDE) (with $F \equiv 0$)

$$\begin{cases} \dot{x}(t) = 0 & t \geq 0 \\ x(t) = \phi(t) & t \in [-h, 0] \end{cases}$$

for given $\phi \in X$. The infinitesimal generator is given by

$$\begin{aligned} D(A_0) &= \{\phi \in C^1([-h,0], Y) : \phi'(0) = 0\} \\ A_0\phi &= \phi' \end{aligned}$$

The sun-dual space $X^{\odot}$ and the semigroup $T_0^{\odot}$

Let $X^{\odot}$ be the subspace of X^* on which T_0^* is strongly continuous:

- $X^{\odot}$ is positively invariant under T_0^*
- $X^{\odot} = \overline{D(A_0^*)}$. In particular it is norm-closed.
- It holds $X^{\odot} = Y^* \times L^1([0,h]; Y^*)$, where the second factor is the space of Bochner integrable Y^*-valued functions on $[0,h]$ [Greiner & Van Neerven, 1992]

Let $T_0^{\odot}$ be the restriction of T_0^* to $X^{\odot}$. Its generator $A_0^{\odot}$ is the part of A_0^* in $X^{\odot}$:

$$D(A_0^{\odot}) = \{\phi^{\odot} \in D(A_0^*) : A_0^* \phi^{\odot} \in X^{\odot}\}$$

Embedding of X in $X^{\odot *}$

X is canonically embedded in $X^{\odot *}$ via

$$j : X \to X^{\odot \star}$$

given by

$$\langle \phi^{\odot}, j\phi \rangle = \langle \phi, \phi^{\odot} \rangle \quad \forall\, \phi \in X, \forall\, \phi^{\odot} \in X^{\odot}$$

where $\langle \phi, \phi^{\odot} \rangle := \phi^{\odot}(\phi)$ (postfix notation).

If $\phi \in C^1([-h,0];Y)$ then $j\phi \in D(A_0^{\odot\star})$ and

$$A_0^{\odot\star} j\phi = (0, \phi') \in X^{\odot\star} = Y^{**} \times (L^1([0,h];Y^*))^*$$

The Abstract Integral Equation

There is a one-to-one correspondence between solutions of ADDE and solutions $u \in C([0,\infty);X)$ of

$$u(t) = T_0(t)\phi + j^{-1}\left(\int_0^t T_0^{\odot\star}(t-s)E(u(s))\,ds\right) \quad \forall\, t \geq 0 \qquad \text{(AIE)}$$

for the nonlinearity $E : X \to X^{\odot\star}$ defined as

$$E(\phi) := (F(\phi), 0)$$

The weak* Riemann integral by definition is the unique $\phi^{\odot\star} \in X^{\odot\star}$ such that

$$\langle \phi^{\odot}, \phi^{\odot\star} \rangle = \int_0^t \langle \phi^{\odot}, T_0^{\odot\star}(t-s)E(u(s)) \rangle \, ds \qquad \forall\, \phi^{\odot} \in X^{\odot}$$

Linearisation at a steady state

Let $L = DG(\hat{\phi}) \in \mathscr{L}(X, Y)$ where $\hat{\phi} \in X$ is an **equilibrium**, i.e.

$$F(\hat{\phi}) = -\alpha\hat{\phi}(0) + G(\hat{\phi}) = 0$$

The solution of the **linearized problem**

$$\begin{cases} \dot{x}(t) = -\alpha x(t) + Lx_t & t \geq 0 \\ x(t) = \phi(t) & t \in [-h, 0] \end{cases}$$

defines a semigroup T on X generated by $A : D(A) \subset X \to X$ where

$$D(A) = \{\phi \in C^1([-h,0], Y) : \phi'(0) = \underbrace{-\alpha\phi(0) + L\phi}_{DF(\hat{\phi})\phi}\}, \quad A\phi = \phi'$$

Since

$$D(A^*) = D(A_0^*),$$

the sun-duals of X with respect to T_0 and T are identical and may both be denoted by $X^\odot$. Let $T^\odot$ be the restriction of T^* to $X^\odot$ and let $A^\odot$ be its generator, then

$$D(A^\odot) = \{\phi^\odot \in D(A^*) : A^*\phi^\odot \in X^\odot\}, \;\; A^\odot = A^*$$

It can be shown that

$$D(A^{\odot\star}) \cap j(X) = D(A_0^{\odot\star}) \cap j(X)$$

It also follows that if $\phi \in C^1([-h,0];Y)$ then $j\phi \in D(A^{\odot\star})$ and

$$A^{\odot\star} j\phi = (0,\phi') + (DF(\hat{\phi})\phi, 0)$$

Finally, all **spectra** coincide: $\sigma(A) = \sigma(A^*) = \sigma(A^\odot) = \sigma(A^{\odot\star})$.

Characterization of the spectrum

For $f \in Y$ and $z \in \mathbb{C}$, let $(\varepsilon_z \otimes f) \in X$ be such that

$$(\varepsilon_z \otimes f)(\theta) = e^{\theta z} f \qquad \forall \theta \in [-h, 0]$$

and

$$L_z \in \mathscr{L}(Y), \qquad L_z f = L(\varepsilon_z \otimes f) \qquad \forall \theta \in [-h, 0]$$

Introduce the **characteristic operator**:

$$\Delta(z) = z + \alpha - L_z \in \mathscr{L}(Y)$$

It holds that $\lambda \in \sigma(A)$ if and only if $0 \in \sigma(\Delta(\lambda))$ and $\psi \in D(A)$ is an eigenvector corresponding to λ if and only if $\psi = \varepsilon_\lambda \otimes q$ where $q \in Y$ satisfies $\Delta(\lambda) q = 0$ [Engel & Nagel, 2000].

For NFEs, the set $\sigma(A) \setminus \{-\alpha\}$ consists of isolated eigenvalues of finite type.

Explicit 1D example

Since $S(0) = 0$, we study stability of $\hat{\phi} \equiv 0$. Let

$$k_i(\lambda) = \lambda + \mu_i \qquad \forall\, i = 1, \dots, N$$

and

$$\mathscr{S} = \{\lambda \in \mathbb{C} : \exists i, j \in \{1, \dots, N\}, i \neq j, \text{ s.t. } k_i^2(\lambda) = k_j^2(\lambda)\}.$$

Define for $\lambda \notin \mathscr{S}$ the **characteristic polynomial**

$$\mathscr{P}(\rho) = \frac{e^{\lambda\tau_0}(\lambda + \alpha)}{2} \prod_{j=1}^{N} (\rho^2 - k_j(\lambda)^2) + \sum_{i=1}^{N} c_i k_i(\lambda) \prod_{\substack{j=1 \\ j \neq i}}^{N} (\rho^2 - k_j(\lambda)^2)$$

and assume that it has $2N$ distinct roots, denoted by $\pm\rho_i(\lambda)$ for $i = 1, 2, \dots, N$.

Spectrum

Under above conditions, introduce

$$S(\lambda) = \begin{bmatrix} S_\lambda^- & S_\lambda^+ \\ S_\lambda^+ & S_\lambda^- \end{bmatrix}$$

where

$$[S_\lambda^-]_{j,i} = \frac{e^{\rho_i(\lambda)}}{\lambda + \mu_j - \rho_i(\lambda)}, \qquad [S_\lambda^+]_{j,i} = \frac{e^{-\rho_i(\lambda)}}{\lambda + \mu_j + \rho_i(\lambda)}$$

Then λ is an eigenvalue of A if and only if $\det S(\lambda) = 0$. The corresponding eigenfunction is $\varepsilon_\lambda \otimes q_\lambda$ with

$$q_\lambda(x) = \sum_{i=1}^{N} \left[\gamma_i e^{\rho_i(\lambda)x} + \gamma_{-i} e^{-\rho_i(\lambda)x}\right] \qquad \forall x \in [-1, 1]$$

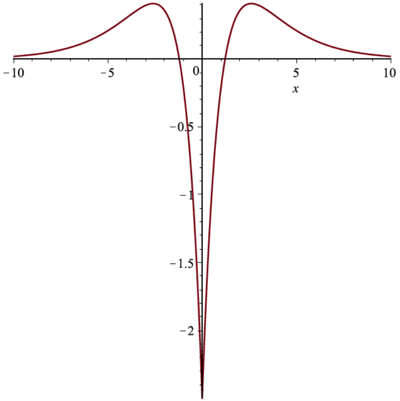
where $\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N, \gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-N}]$ is a solution to $S(\lambda)\Gamma = 0$.

Example: Inverse "wizard hat" connectivity

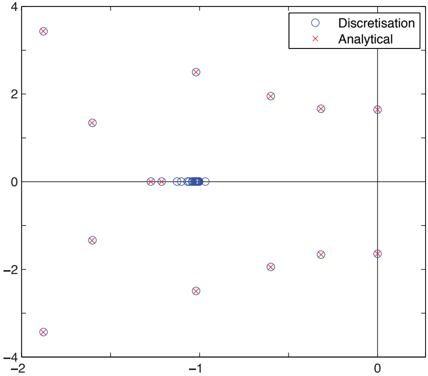
Take $\alpha = \tau_0 = 1$ and

$$J(x,x') = \hat{c}_1 e^{-\mu_1|x-x'|} + \hat{c}_2 e^{-\mu_2|x-x'|} \qquad \forall\, x, x' \in [-1,1]$$

with $\hat{c}_1 = 3,\ \hat{c}_2 = -5.5,\ \mu_1 = 0.5,\ \mu_2 = 1$. Since $S'(0) = \frac{r}{4}$, we have $c_i = \frac{r}{4}\hat{c}_i$.



Hopf bifurcation at $r = 4.220215$



The approximate eigenvalues were computed with DDE-BIFTOOL [Engelborghs et al., 2002].

Center Manifold for NFEs

Suppose that A has $n_c \geq 1$ critical eigenvalues with $\Re(\lambda) = 0$. This implies the existence of an invariant **center manifold** $\mathcal{W}^c_{loc}$ on which

$$\dot{u}(t) = j^{-1}\left(A^{\odot\star} j u(t) + R(u(t))\right) \qquad \forall\, t \in \mathbb{R}$$

where $R : X \to X^{\odot\star}$ is

$$R(\phi) = E(\phi) - DE(\hat{\phi})\phi = \frac{1}{2}B(\phi,\phi) + \frac{1}{3!}C(\phi,\phi,\phi) + O(\|\phi\|^4)$$

For NFEs, we can apply the finite-dimensional approach, taking into account that for $\lambda \in \mathbb{C} \setminus \{-\alpha\}$ the linear equation

$$(\lambda - A^{\odot\star})\phi^{\odot\star} = \psi^{\odot\star}$$

is solvable for $\phi^{\odot\star} \in D(A^{\odot\star})$ given $\psi^{\odot\star} \in X^{\odot\star}$ if and only if $\langle \phi^{\odot}, \psi^{\odot\star}\rangle = 0$ for all $\phi^{\odot} \in \mathcal{N}(\lambda - A^*)$ (**Fredholm Solvability**).

Andronov-Hopf bifurcation in NFEs

Let ϕ and $\phi^{\odot}$ be complex eigenvectors of A and A^* corresponding to $\lambda_1 = i\omega_0$,

$$A\phi = i\omega_0\phi, \;\; A^*\phi^{\odot} = i\omega_0\phi^{\odot}, \;\; \omega_0 > 0,$$

and satisfying $\langle \phi, \phi^{\odot} \rangle = 1$.

- The projection of $u(t) \in \mathscr{W}^c_{loc}$ onto the tangent space to $\mathscr{W}^c_{loc}$ at $\hat{\phi}$ satisfies

$$\dot{z} = i\omega_0 z + c_1 z|z|^2 + O(|z|^4), \;\; z \in \mathbb{C}$$

- Center manifold representation:

$$u = \mathscr{H}(z, \bar{z}) = z\phi + \bar{z}\bar{\phi} + \sum_{2 \le j+k \le 3} \frac{1}{j!k!} h_{jk} z^j \bar{z}^k + O(|z|^4)$$

The invariance of $\mathscr{W}^c_{loc}$ implies the **homological equation**

$$A^{\odot\star} j\mathscr{H}(z,\overline{z}) + R(\mathscr{H}(z,\overline{z})) = j\left(D_z\mathscr{H}(z,\overline{z})\dot{z} + D_{\overline{z}}\mathscr{H}(z,\overline{z})\dot{\overline{z}}\right)$$

that gives

$$\begin{cases} -A^{\odot\star}\, jh_{20} = B(\phi,\overline{\phi}) \\ (2i\omega_0 - A^{\odot\star})\, jh_{11} = B(\phi,\phi) \end{cases} \implies \begin{cases} jh_{20} = R(0,A^{\odot\star})B(\phi,\overline{\phi}) \\ jh_{11} = R(2i\omega, A^{\odot\star})B(\phi,\phi) \end{cases}$$

as well as

$$(i\omega_0 I - A^{\odot\star})jh_{21} = C(\phi,\phi,\overline{\phi}) + B(\overline{\phi},h_{20}) + 2B(\phi,h_{11}) - 2c_1 j\phi$$

so that the Fredholm Solvability implies

$$c_1 = \frac{1}{2}\langle\phi^{\odot}, C(\phi,\phi,\overline{\phi}) + B(\overline{\phi},h_{20}) + 2B(\phi,h_{11})\rangle$$

and

$$L_1 = \frac{1}{\omega_0}\Re(c_1)$$

Computation of resolvents

To compute $\psi^{\odot\star} = R(z, A^{\odot\star})(y, 0)$, we need to solve

$$(z - A^{\odot\star})\psi^{\odot\star} = (y, 0)$$

where $z \in \rho(A)$, $y \in Y$ and $\psi^{\odot\star} \in D(A^{\odot\star})$.

For each $y \in Y$ the function $\psi = \varepsilon_z \otimes \Delta(z)^{-1} y$ is the unique solution in $C^1([-h, 0]; Y)$ of the system

$$\begin{cases} z\psi(0) - DF(\hat{\phi})\psi = y \\ z\psi - \psi' = 0 \end{cases}$$

Then

$$\psi^{\odot\star} = j\psi = \begin{bmatrix} \Delta(z)^{-1} y \\ \varepsilon_z \otimes \Delta(z)^{-1} y \end{bmatrix}$$

Evaluation of pairings

Let $P^{\odot}$ and $P^{\odot\star}$ be the spectral projections on $X^{\odot}$ and $X^{\odot\star}$ corresponding to a simple $\lambda \in \sigma(A)$.
We want to evaluate $\langle \phi^{\odot}, \phi^{\odot\star} \rangle$ where

$$\phi^{\odot\star} = (y, 0) \in Y \times \{0\} \subset X^{\odot\star}$$

Since the range of $P^{\odot\star}$ is spanned by $j\phi$ we have

$$P^{\odot\star}\phi^{\odot\star} = \kappa j\phi$$

for a certain $\kappa \in \mathbb{C}$. Furthermore,

$$\langle \phi^{\odot}, \phi^{\odot\star} \rangle = \langle P^{\odot}\phi^{\odot}, \phi^{\odot\star} \rangle = \langle \phi^{\odot}, P^{\odot\star}\phi^{\odot\star} \rangle = \kappa \langle \phi^{\odot}, j\phi \rangle = \kappa$$

On the other hand [Dunford & Schwartz, 1958]

$$P^{\odot\star}\phi^{\odot\star} = \frac{1}{2\pi i}\oint_{\partial C_\lambda} R(z, A^{\odot\star})\phi^{\odot\star}\,dz = \kappa j\phi$$

and the first component shows that κ can be found from

$$\frac{1}{2\pi i}\oint_{\partial C_\lambda} \Delta(z)^{-1} y\,dz = \kappa\phi(0)$$

For the explicit 1D example, the computation of pairings can be reduced further.

Explicit 1D example

$$\phi(t,x) = e^{\lambda t}\left[\gamma_1(e^{\rho_1 x} + e^{-\rho_1 x}) + \gamma_2(e^{\rho_2 x} + e^{-\rho_2 x})\right] \qquad \forall\, t \in [-h, 0]$$

where

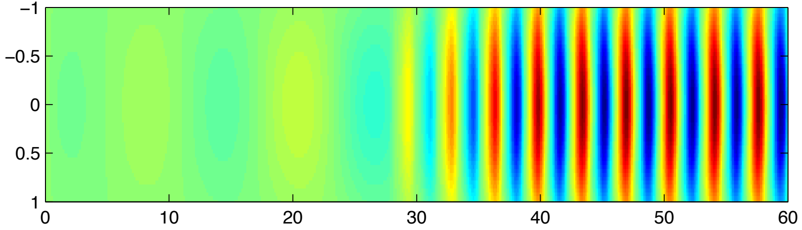
$$\rho_1 = 0.321607348361597 - 0.880461478656249i$$
$$\rho_2 = 0.110838003673357 - 2.312123026384049i$$
$$\gamma_1 = -0.191821747840362 - 0.172140605861736i$$
$$\gamma_2 = -0.080160108888561$$

corresponding to $\lambda = i\omega_0 = 1.644003102046893i$.

$$c_1 = \frac{1}{2}\langle \phi^{\odot}, C(\phi,\phi,\overline{\phi})\rangle \approx -0.326 + 0.0389i$$

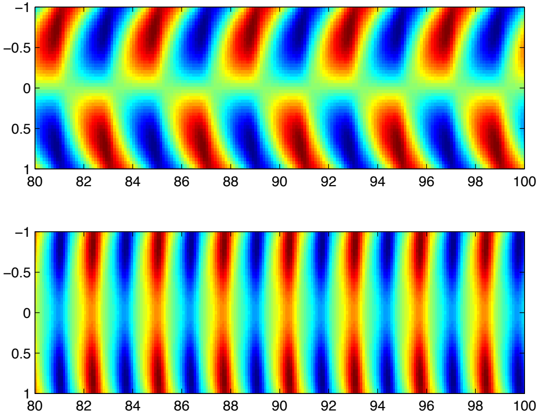
Simulations after Hopf bifurcation

Thus, the first Lyapunov coefficient is $L_1 = \frac{1}{\omega_0}\Re(c_1) \approx -0.198 < 0$ indicating a **supercritical** Hopf bifurcation.



Forward time simulation of discretized system ($m = 50$) for $r = 6$ beyond Hopf bifurcation. A long transient is observed before the solution approaches the stable limit cycle.

Double Hopf bifurcation (no Chaos)



Bi-stability near the double Hopf bifurcation: for $r = 6$ and $\mu_2 = 1$ the time evolution is shown for different initial conditions ($m = 50$).

four normal form coefficients needed

Contents

Future directions

- Spatially extended neurons as extensions of neural fields.
- Extend the explicit spectral analysis to more dimensions.
- Include extracellular dynamics.
- Extend the theory to abstract semilinear delay differential equations of the form

$$\begin{aligned} \dot{x}(t) &= Bx(t) + F(x_t) \qquad t \geq 0 \\ x(t) &= \phi(t) \qquad\qquad\quad t \in [-h, 0] \end{aligned}$$

where $B\colon\ D(B) \subseteq Y \mapsto Y$ is the generator of a C_0-semigroup S on Y.