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1. BIFURCATIONS AND THEIR CLASSIFICATION

Consider a smooth 2D system depending on one parameter

\[ \dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \ \alpha \in \mathbb{R}. \]

**Definition 1** A point \( \alpha_0 \) is called a **bifurcation point** if in any neighborhood of \( \alpha_0 \) there is a point \( \alpha \) for which

\[ \dot{X} = f(X, \alpha) \not\sim \dot{X} = f(X, \alpha_0). \]

The appearance of a topologically non-equivalent system is called a **bifurcation**.

Since the number of equilibria, the number of periodic orbits, and their stability, as well as the presence of connecting orbits, are topological invariants, a bifurcation of the 2D-system means a change of (some of) these properties.
**Definition 2** A codimension of a bifurcation is the number of conditions on which the bifurcating phase object has to satisfy.

**Classification of codimension-one bifurcations:**

1. **Local (near equilibria)**
   - *saddle-node (fold)*
   - *(Andronov–)* *Hopf*

2. **Local of cycles (near periodic orbits)**
   - *(cyclic) fold*

3. **Bifurcations of homoclinic and heteroclinic orbits**
   - *saddle homoclinic*
   - *saddle–node homoclinic*
   - *heteroclinic*

Only codim 1 bifurcations occur in generic one-parameter systems.
2. (LOCAL) BIFURCATIONS OF EQUILIBRIA

- If $X_0$ is a hyperbolic equilibrium of $\dot{X} = f(X, \alpha_0)$, then it remains hyperbolic for all $\alpha$ sufficiently close to $\alpha_0$ (but can slightly shift).

- A local bifurcation can happen only to a non-hyperbolic equilibrium with $\Re(\lambda) = 0$.

- Generic codimension-1 critical cases:
  
  1. **Fold (saddle-node):** $\lambda_1 = 0$ ($\lambda_2 \neq 0, \ a \neq 0$)
     
     \[
     \begin{align*}
     \dot{x} &= ax^2, \\
     \dot{y} &= \lambda_2 y.
     \end{align*}
     \]

  2. **Andronov-Hopf (weak focus):** $\lambda_{1,2} = \pm i\omega$ ($\omega > 0, \ l_1 \neq 0$)
     
     \[
     \begin{align*}
     \dot{\rho} &= l_1\rho^3, \\
     \dot{\phi} &= 1.
     \end{align*}
     \]
**Fold:** \( \lambda_1 = 0 \)

**Theorem 1**  If \( a \neq 0 \) and \( \lambda_2 \neq 0 \), then \( \dot{X} = f(X, \alpha) \) is locally topologically equivalent near the saddle-node to

\[
\begin{align*}
\dot{x} &= \beta(\alpha) + ax^2, \\
\dot{y} &= \lambda_2 y,
\end{align*}
\]

where \( \beta(0) = 0 \).

Two equilibria \( O_{1,2} = \left( \mp \sqrt{-\frac{\beta}{a}}, 0 \right) \) collide and disappear in the 1D center manifold \( W^c = \{y = 0\} \), provided \( \beta'(0) \neq 0 \).
Andronov-Hopf: \( \lambda_{1,2} = \pm i\omega \)

**Theorem 2** If \( l_1 \neq 0 \) and \( \omega > 0 \), then \( \dot{X} = f(X, \alpha) \) is locally topologically equivalent near the weak focus to

\[
\begin{align*}
\dot{\rho} &= \rho(\beta(\alpha) + l_1 \rho^2), \\
\dot{\phi} &= 1.
\end{align*}
\]

where \( \beta(0) = 0 \).

A limit cycle \( \rho_0 = \sqrt{-\frac{\beta}{l_1}} > 0 \) appears while the focus changes stability.

The direction of the cycle bifurcation is determined by the first Lyapunov coefficient \( l_1 \) of the weak focus:

- **supercritical** (soft, non-catastrophic) Andronov-Hopf bifurcation \( (l_1 < 0) \);
- **subcritical** (hard, catastrophic) Andronov-Hopf bifurcation \( (l_1 > 0) \).
Supercritical Andronov-Hopf bifurcation: $l_1 < 0$

The stable equilibrium is replaced by small-amplitude oscillations within an attracting domain.
Subcritical Andronov-Hopf bifurcation: $l_1 > 0$

The domain of attraction of the stable focus shrinks, while it becomes unstable.
Example: \[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -x + \alpha y + x^2 + xy + y^2.
\end{aligned}
\]

At \(\alpha = 0\) the equilibrium \(x = y = 0\) of the reversed system

\[
\begin{aligned}
\dot{x} &= -y, \\
\dot{y} &= x - x^2 - xy - y^2,
\end{aligned}
\]

has eigenvalues \(\lambda_{1,2} = \pm i\) \((\omega = 1)\).

Introduce \(z = x + iy\), then \(x^2 + y^2 = |z|^2 = zz\) and

\[
\begin{aligned}
\dot{z} &= x + iy = -y + ix - ix^2 - ixy - iy^2 \\
&= i(z - iz\bar{z} - \frac{1}{4}(z^2 - \bar{z}^2)) = iz - \frac{1}{4}z^2 - iz\bar{z} + \frac{1}{4}\bar{z}^2
\end{aligned}
\]

so that \(\omega = 1\), \(g_{20} = -\frac{1}{2}\), \(g_{11} = -i\), \(g_{02} = \frac{1}{2}\), \(g_{21} = 0\).

\[
\tilde{l}_1 = \frac{1}{2\omega^2} \Re(i g_{20} g_{11} + \omega g_{21}) = \frac{1}{2} \left(i \frac{1}{2} i + 1 \cdot 0\right) = -\frac{1}{4}.
\]

For the original system, \(l_1 = \frac{1}{4} > 0 \Rightarrow\) subcritical Hopf bifurcation (an unstable cycle exists for small \(\alpha < 0\) but disappears for \(\alpha > 0\))
Practical computation of $a$ and $l_1$ in $\mathbb{R}^2 (n = 2)$

Suppose $X_0 = 0$, $\alpha_0 = 0$ and write the Taylor expansion in the original coordinates:

$$f(X, 0) = AX + \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X) + O(4)$$

where

$$(AX)_i = \sum_{j=1}^{n} \frac{\partial f_i(U, 0)}{\partial U_j} \bigg|_{U=0} X_j,$$

$$B_i(X, Y) = \sum_{j,k=1}^{n} \frac{\partial^2 f_i(U, 0)}{\partial U_j \partial U_k} \bigg|_{U=0} X_j Y_k,$$

$$C_i(X, Y, Z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 f_i(U, 0)}{\partial U_j \partial U_k \partial U_l} \bigg|_{U=0} X_j Y_k Z_l,$$

for $i = 1, \ldots, n$. 
Theorem 3  The fold normal form coefficient can be computed as

\[ a = \frac{1}{2} \langle p, B(q, q) \rangle \]

where \( p, q \in \mathbb{R}^2 \) satisfy

\[ Aq = A^T p = 0 \]

and \( p^T q \equiv \langle p, q \rangle = 1 \).

Theorem 4  The first Lyapunov coefficient can be computed in 2D as

\[ l_1 = \frac{1}{2\omega^2} \Re \left[ i \langle p, B(q, q) \rangle \langle p, B(q, \bar{q}) \rangle + \omega \langle p, C(q, q, \bar{q}) \rangle \right] \]

where \( p, q \in \mathbb{C}^2 \) satisfy

\[ Aq = i\omega q, \quad A^T p = -i\omega p \]

and \( \bar{p}^T q \equiv \langle p, q \rangle = 1 \).
Example: Hopf bifurcation in a prey-predator system

Consider the following system

\[
\begin{aligned}
\dot{x}_1 &= rx_1(1-x_1) - \frac{cx_1x_2}{\alpha + x_1} \\
\dot{x}_2 &= -dx_2 + \frac{cx_1x_2}{\alpha + x_1}
\end{aligned}
\]

\[
\begin{aligned}
\dot{x}_1 &= rx_1(\alpha + x_1)(1-x_1) - cx_1x_2 \\
\dot{x}_2 &= -\alpha dx_2 + (c-d)x_1x_2
\end{aligned}
\]

At \( \alpha_0 = \frac{c-d}{c+d} \) the last system has the equilibrium \((x_1^{(0)}, x_2^{(0)}) = \left( \frac{d}{c+d}, \frac{rc}{(c+d)^2} \right)\) with eigenvalues \( \lambda_{1,2} = \pm i\omega \), where \( \omega^2 = \frac{rc^2d(c-d)}{(c+d)^3} > 0. \)

Translate the origin of the coordinates to this equilibrium by

\[
\begin{aligned}
\begin{cases}
\dot{x}_1 &= x_1^{(0)} + X_1, \\
\dot{x}_2 &= x_2^{(0)} + X_2.
\end{cases}
\end{aligned}
\]
This transforms the system into

\[
\begin{align*}
\dot{X}_1 &= -\frac{cd}{c+d}X_2 - \frac{rd}{c+d}X_1^2 - cX_1X_2 - rX_1^3, \\
\dot{X}_2 &= \frac{rc(c-d)}{(c+d)^2}X_1 + (c-d)X_1X_2,
\end{align*}
\]

that can be represented as

\[
\dot{X} = AX + \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X),
\]

where

\[
A = \begin{pmatrix}
0 & -\frac{cd}{c+d} \\
\omega^2(c+d) & 0 \\
cd & 0 \\
\end{pmatrix}, \quad B(X, Y) = \begin{pmatrix}
-\frac{2rd}{c+d}X_1Y_1 - c(X_1Y_2 + X_2Y_1) \\
(c-d)(X_1Y_2 + X_2Y_1)
\end{pmatrix}
\]

and

\[
C(X, Y, Z) = \begin{pmatrix}
-6rX_1Y_1Z_1 \\
0
\end{pmatrix}.
\]
The complex vectors
\[ q = \begin{pmatrix} cd \\ -i\omega(c + d) \end{pmatrix}, \quad p = \frac{1}{2\omega cd(c + d)} \begin{pmatrix} \omega(c + d) \\ -icd \end{pmatrix}. \]
satisfy \( Aq = i\omega q, \) \( A^T p = -i\omega p \) and \( \langle p, q \rangle = 1. \)

Then
\[ g_{20} = \langle p, B(q, q) \rangle = \frac{cd(c^2 - d^2 - rd) + i\omega c(c + d)^2}{(c + d)}, \]
\[ g_{11} = \langle p, B(q, \bar{q}) \rangle = -\frac{rcd^2}{(c + d)}, \quad g_{21} = \langle p, C(q, q, \bar{q}) \rangle = -3rc^2d^2, \]
and the first Lyapunov coefficient
\[ l_1(\alpha_0) = \frac{1}{2\omega^2} \text{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rc^2d^2}{\omega} < 0. \]
Therefore, a **stable cycle** bifurcates from the equilibrium via the super-critical Hopf bifurcation for $\alpha < \alpha_0$.

One can prove that the cycle is **unique**.
3. LOCAL BIFURCATION OF CYCLES: \( \mu = 1 \)

Parameter-dependent Poincaré map:

\[
\xi \mapsto \tilde{\xi} = P(\xi, \alpha),
\]

where \( P(\xi, 0) = \xi + O(2) \) \( (\mu = 1) \)

**Lemma 1** If

\[
p_2(0) = \frac{1}{2} P_{\xi\xi}(0, 0) \neq 0,
\]

then there exists a smooth function \( \delta = \delta(\alpha) \) such that the substitution \( x = \xi + \delta(\alpha) \) reduces the map

\[
\xi \mapsto P(\xi, \alpha) = p_0(\alpha) + [1 + g(\alpha)]\xi + p_2(\alpha)\xi^2 + O(3),
\]

where \( g(0) = 0, p_0(0) = P(0, 0) = 0, \) to the form

\[
x \mapsto \tilde{x} = \beta(\alpha) + x + b(\alpha)x^2 + O(3)
\]

with \( \beta(0) = 0 \) and \( b(0) = p_2(0) \neq 0. \)
Cyclic fold: \( x \mapsto \beta + x + bx^2, \ b > 0 \)

Two hyperbolic cycles (unstable \( C_1 \) and stable \( C_2 \)) collide forming a non-hyperbolic cycle \( C_0 \), and disappear.
4. (GLOBAL) BIFURCATIONS OF CONNECTING ORBITS

- Saddle homoclinic bifurcation

**Singular map:** \[ \eta \mapsto \xi = \eta \frac{-\lambda_1}{\lambda_2}. \]

**Regular map:**
\[ \xi \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\xi + O(2), \quad A(0) > 0. \]

**Poincaré map:**
\[ \eta \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\eta \frac{-\lambda_1}{\lambda_2} + \ldots \]
Saddle homoclinic bifurcation: $\sigma < 0$

A stable cycle $C_\beta$ bifurcates from $\Gamma_0$ while the separatrices exchange.
Saddle homoclinic bifurcation: $\sigma > 0$

An unstable cycle $C_\beta$ bifurcates from $\Gamma_0$ while the separatrices exchange.

**Theorem 5 (Melnikov)**

$$\beta'(0) \neq 0 \iff \int_0^\infty \exp \left( - \int_0^t \text{div } f(X^0(s))ds \right) \left( f_1 \frac{\partial f_2}{\partial \alpha} - f_2 \frac{\partial f_1}{\partial \alpha} \right) (X^0(t))dt \neq 0$$
- Homoclinic saddle-node bifurcation:

- Heteroclinic saddle bifurcation:
Example: Allee effect in a prey-predator system

\[\begin{align*}
\dot{x} &= x(x - l)(1 - x) - xy, \\
\dot{y} &= -\gamma y(m - x).
\end{align*}\]
Remarks:

1. There are **no** other codim 1 bifurcations in generic smooth 2D ODEs.

2. Heteroclinic bifurcation points can **accumulate**: