BIFURCATION PHENOMENA

Lecture 2: One-parameter bifurcations of planar ODEs

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Contents

Literature

1. A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.G. Maier *Theory of Bifurcations of Dynamic Systems on a Plane*, Willey & Sons, London, 1973

2. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, and L.O. Chua *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, Singapore, 2001

3. Yu.A. Kuznetsov *Elements of Applied Bifurcation Theory*, 3rd ed. Applied Mathematical Sciences 112, Springer-Verlag, New York, 2004

1. BIFURCATIONS AND THEIR CLASSIFICATION

Consider a smooth 2D system depending on one parameter

$$\dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \ \alpha \in \mathbb{R}.$$

Definition 1 *A point α_0 is called a* **bifurcation point** *if in any neighborhood of α_0 there is a point α for which*

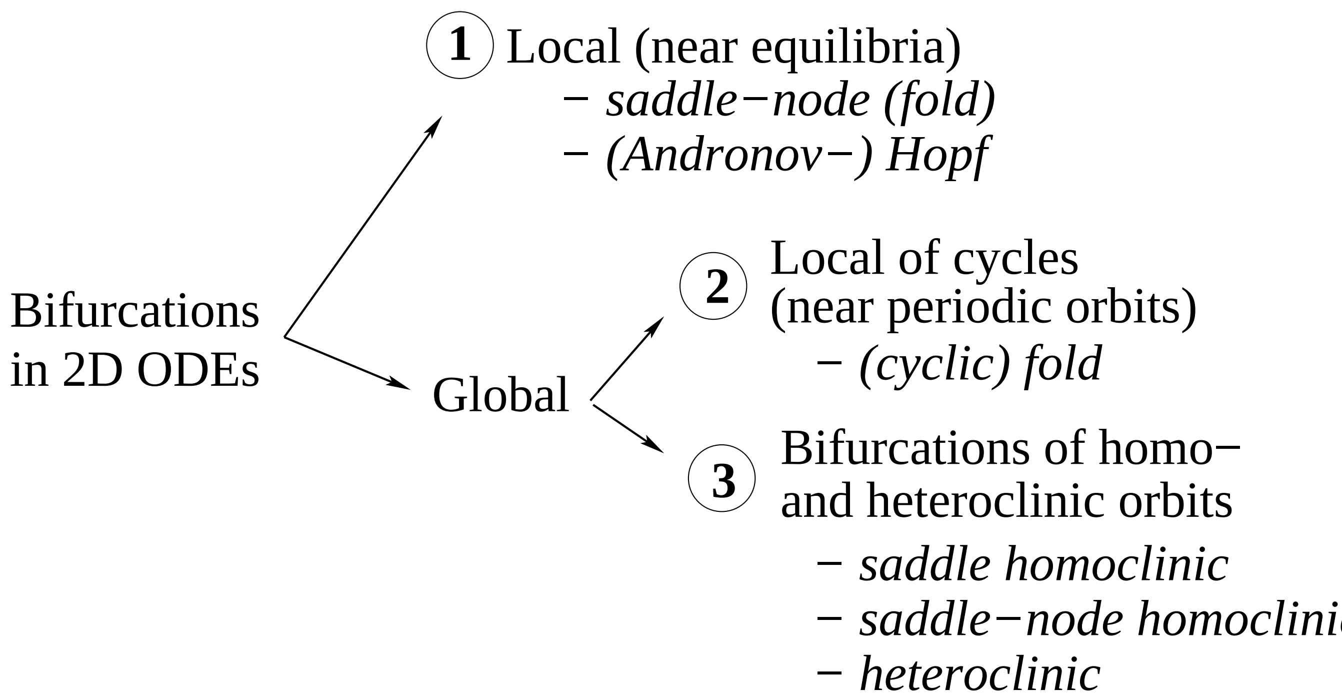
$$\dot{X} = f(X, \alpha) \quad \not\sim \quad \dot{X} = f(X, \alpha_0).$$

The appearance of a topologically non-equivalent system is called a **bifurcation**.

Since the number of equilibria, the number of periodic orbits, and their stability, as well as the presence of connecting orbits, are topological invariants, a bifurcation of the 2D-system means a change of (some of) these properties.

Definition 2 *A* **codimension** *of a bifurcation is the number of conditions on which the bifurcating phase object has to satisfy.*

Classification of codimension-one bifurcations:



Only codim 1 bifurcations occur in generic one-parameter systems.

2. (LOCAL) BIFURCATIONS OF EQUILIBRIA

- If X_0 is a hyperbolic equilibrium of $\dot{X} = f(X, \alpha_0)$, then it remains hyperbolic for all α sufficiently close to α_0 (but can slightly shift).

- A local bifurcation can happen only to a non-hyperbolic equilibrium with $\Re(\lambda) = 0$.

- Generic codimension-1 critical cases:

  1. **Fold (saddle-node)**: $\lambda_1 = 0$ $(\lambda_2 \neq 0,\ a \neq 0)$

$$\begin{cases} \dot{x} &= ax^2, \\ \dot{y} &= \lambda_2 y. \end{cases}$$

  2. **Andronov-Hopf (weak focus)**: $\lambda_{1,2} = \pm i\omega$ $(\omega > 0,\ l_1 \neq 0)$

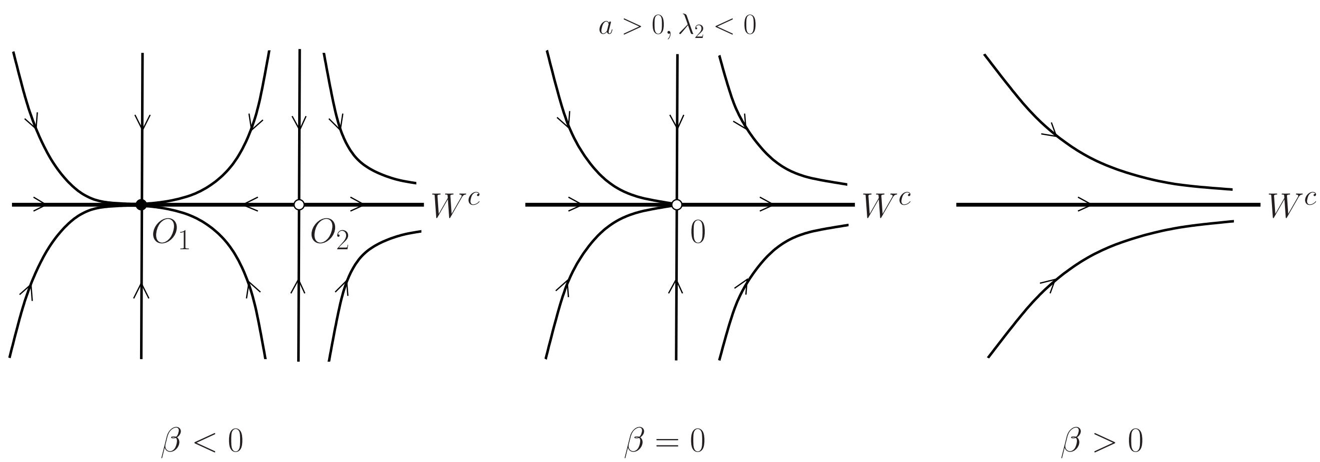
$$\begin{cases} \dot{\rho} &= l_1 \rho^3, \\ \dot{\varphi} &= 1. \end{cases}$$

Fold: $\lambda_1 = 0$

Theorem 1 *If $a \neq 0$ and $\lambda_2 \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near the saddle-node to*

$$\begin{cases} \dot{x} &= \beta(\alpha) + ax^2, \\ \dot{y} &= \lambda_2 y, \end{cases}$$

where $\beta(0) = 0$.



Two equilibria $O_{1,2} = \left(\mp\sqrt{\frac{-\beta}{a}}, 0 \right)$ collide and disappear in the 1D center manifold $W^c = \{y = 0\}$, provided $\beta'(0) \neq 0$.

Andronov-Hopf: $\lambda_{1,2} = \pm i\omega$

Theorem 2 *If* $l_1 \neq 0$ *and* $\omega > 0$*, then* $\dot{X} = f(X, \alpha)$ *is locally topologically equivalent near the weak focus to*

$$\begin{cases} \dot{\rho} &= \rho(\beta(\alpha) + l_1\rho^2), \\ \dot{\varphi} &= 1. \end{cases}$$

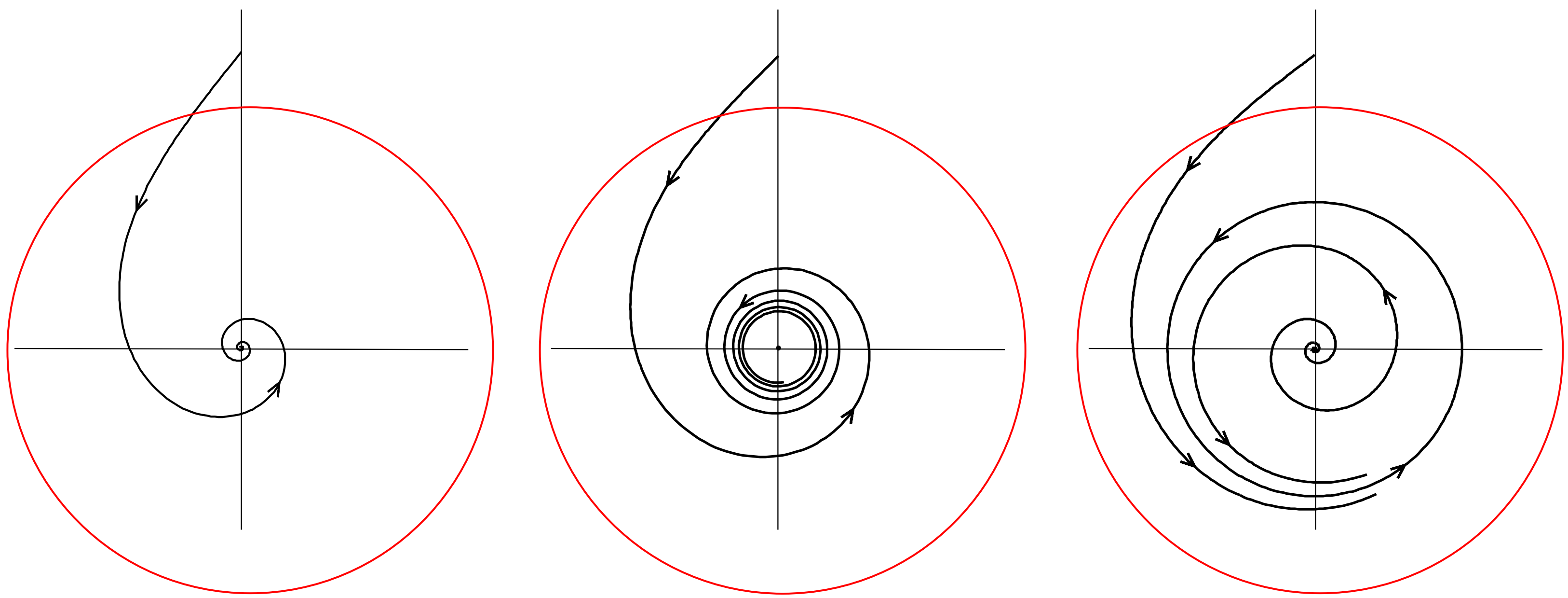
where $\beta(0) = 0$*.*

A **limit cycle** $\rho_0 = \sqrt{\frac{-\beta}{l_1}} > 0$ appears while the focus changes stability.

The direction of the cycle bifurcation is determined by the **first Lyapunov coefficient** l_1 of the weak focus:

- **supercritical** (soft, non-catastrophic) Andronov-Hopf bifurcation ($l_1 < 0$);
- **subcritical** (hard, catastrophic) Andronov-Hopf bifurcation ($l_1 > 0$).

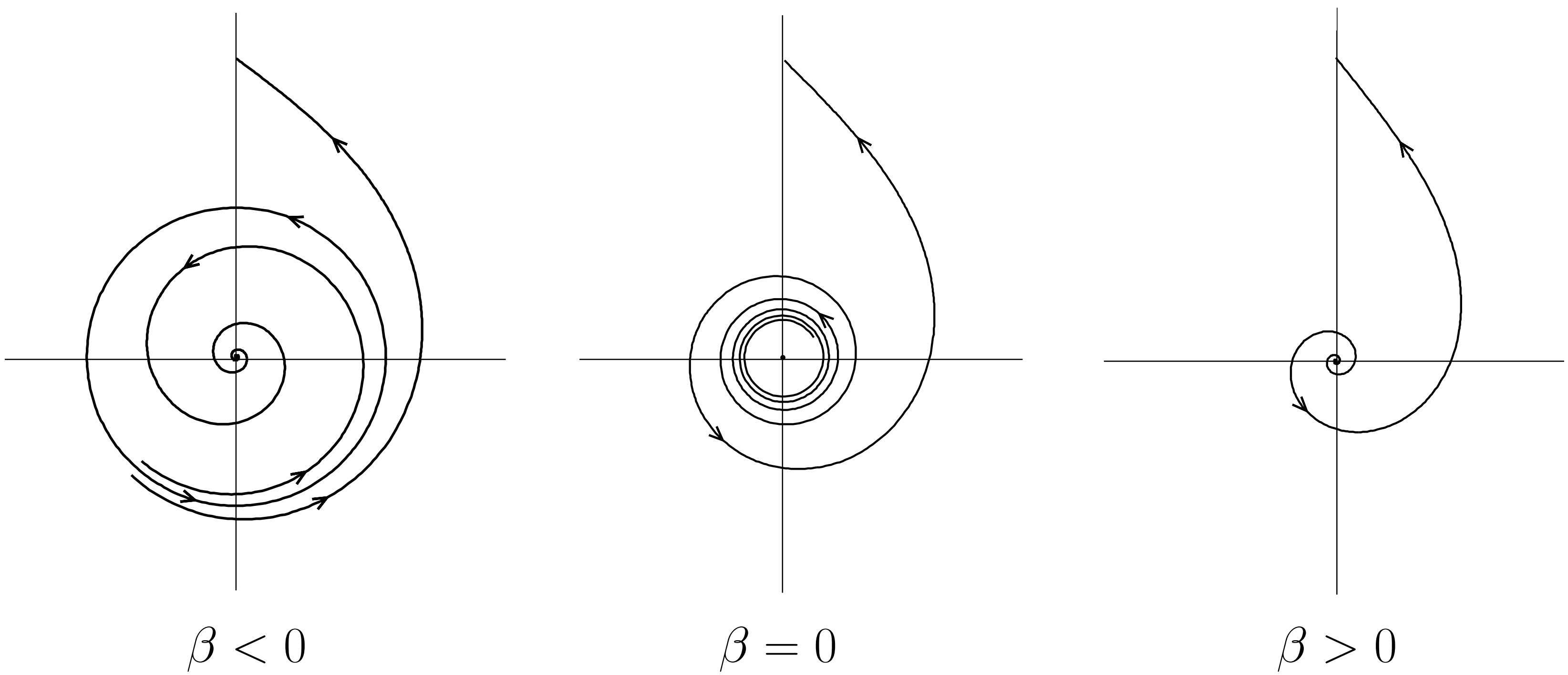
Supercritical Andronov-Hopf bifurcation: $l_1 < 0$



$\beta < 0$ $\beta = 0$ $\beta > 0$

The stable equilibrium is replaced by small-amplitude oscillations within an attracting domain.

Subcritical Andronov-Hopf bifurcation: $l_1 > 0$



The domain of attraction of the stable focus shrinks, while it becomes unstable.

Example: $\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + \alpha y + x^2 + xy + y^2. \end{cases}$

At $\alpha = 0$ the equilibrium $x = y = 0$ of the **reversed system**

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x - x^2 - xy - y^2, \end{cases}$$

has eigenvalues $\lambda_{1,2} = \pm i$ $(\omega = 1)$.

Introduce $z = x + iy$, then $x^2 + y^2 = |z|^2 = z\bar{z}$ and

$$\begin{aligned} \dot{z} &= \dot{x} + i\dot{y} = -y + ix - ix^2 - ixy - iy^2 \\ &= iz - iz\bar{z} - \frac{1}{4}(z^2 - \bar{z}^2) = iz - \frac{1}{4}z^2 - iz\bar{z} + \frac{1}{4}\bar{z}^2 \end{aligned}$$

so that $\omega = 1,\ g_{20} = -\frac{1}{2},\ g_{11} = -i,\ g_{02} = \frac{1}{2},\ g_{21} = 0.$

$$\tilde{l}_1 = \frac{1}{2\omega^2}\Re(ig_{20}g_{11} + \omega g_{21}) = \frac{1}{2}\left(i\ \frac{1}{2}\ i + 1 \cdot 0\right) = -\frac{1}{4}.$$

For the original system, $l_1 = \frac{1}{4} > 0 \ \Rightarrow$ subcritical Hopf bifurcation (an **unstable cycle** exists for small $\alpha < 0$ but disappears for $\alpha > 0$)

Practical computation of a and l_1 in $\mathbb{R}^2$ $(n = 2)$

Suppose $X_0 = 0,\ \alpha_0 = 0$ and write the Taylor expansion in the original coordinates:

$$f(X,0) = AX + \frac{1}{2}B(X,X) + \frac{1}{6}C(X,X,X) + O(4)$$

where

$$\begin{aligned}
(AX)_i &= \sum_{j=1}^{n} \left.\frac{\partial f_i(U,0)}{\partial U_j}\right|_{U=0} X_j, \\
B_i(X,Y) &= \sum_{j,k=1}^{n} \left.\frac{\partial^2 f_i(U,0)}{\partial U_j \partial U_k}\right|_{U=0} X_j Y_k, \\
C_i(X,Y,Z) &= \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 f_i(U,0)}{\partial U_j \partial U_k \partial U_l}\right|_{U=0} X_j Y_k Z_l,
\end{aligned}$$

for $i = 1, \ldots, n$.

Theorem 3 *The fold normal form coefficient can be computed as*

$$a = \frac{1}{2}\langle p, B(q,q)\rangle$$

where $p, q \in \mathbb{R}^2$ *satisfy*

$$Aq = A^{\mathsf{T}} p = 0$$

and $p^{\mathsf{T}} q \equiv \langle p, q\rangle = 1$.

Theorem 4 *The first Lyapunov coefficient can be computed in 2D as*

$$l_1 = \frac{1}{2\omega^2}\Re\left[i\langle p, B(q,q)\rangle\langle p, B(q,\bar{q})\rangle + \omega\langle p, C(q,q,\bar{q})\rangle\right]$$

where $p, q \in \mathbb{C}^2$ *satisfy*

$$Aq = i\omega q, \quad A^{\mathsf{T}} p = -i\omega p$$

and $\bar{p}^{\mathsf{T}} q \equiv \langle p, q\rangle = 1$.

Example: Hopf bifurcation in a prey-predator system

Consider the following system

$$\begin{cases} \dot{x}_1 &= rx_1(1-x_1) - \dfrac{cx_1x_2}{\alpha + x_1} \\ \dot{x}_2 &= -dx_2 + \dfrac{cx_1x_2}{\alpha + x_1} \end{cases} \sim \begin{cases} \dot{x}_1 &= rx_1(\alpha + x_1)(1-x_1) - cx_1x_2 \\ \dot{x}_2 &= -\alpha dx_2 + (c-d)x_1x_2 \end{cases}$$

At $\alpha_0 = \frac{c-d}{c+d}$ the last system has the equilibrium $\left(x_1^{(0)}, x_2^{(0)}\right) = \left(\frac{d}{c+d}, \frac{rc}{(c+d)^2}\right)$ with eigenvalues $\lambda_{1,2} = \pm i\omega$, where $\omega^2 = \frac{rc^2d(c-d)}{(c+d)^3} > 0$.

Translate the origin of the coordinates to this equilibrium by

$$\begin{cases} x_1 &= x_1^{(0)} + X_1, \\ x_2 &= x_2^{(0)} + X_2. \end{cases}$$

This transforms the system into

$$\begin{cases} \dot{X}_1 &=& -\dfrac{cd}{c+d}X_2 - \dfrac{rd}{c+d}X_1^2 - cX_1X_2 - rX_1^3, \\ \dot{X}_2 &=& \dfrac{rc(c-d)}{(c+d)^2}X_1 + (c-d)X_1X_2, \end{cases}$$

that can be represented as

$$\dot{X} = AX + \frac{1}{2}B(X,X) + \frac{1}{6}C(X,X,X),$$

where

$$A = \begin{pmatrix} 0 & -\dfrac{cd}{c+d} \\ \dfrac{\omega^2(c+d)}{cd} & 0 \end{pmatrix}, \; B(X,Y) = \begin{pmatrix} -\dfrac{2rd}{c+d}X_1Y_1 - c(X_1Y_2 + X_2Y_1) \\ (c-d)(X_1Y_2 + X_2Y_1) \end{pmatrix}$$

and

$$C(X,Y,Z) = \begin{pmatrix} -6rX_1Y_1Z_1 \\ 0 \end{pmatrix}.$$

The complex vectors

$$q = \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix}, \quad p = \frac{1}{2\omega cd(c+d)} \begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix}.$$

satisfy $Aq = i\omega q,\ A^{\mathsf{T}} p = -i\omega p$ and $\langle p, q \rangle = 1$.

Then

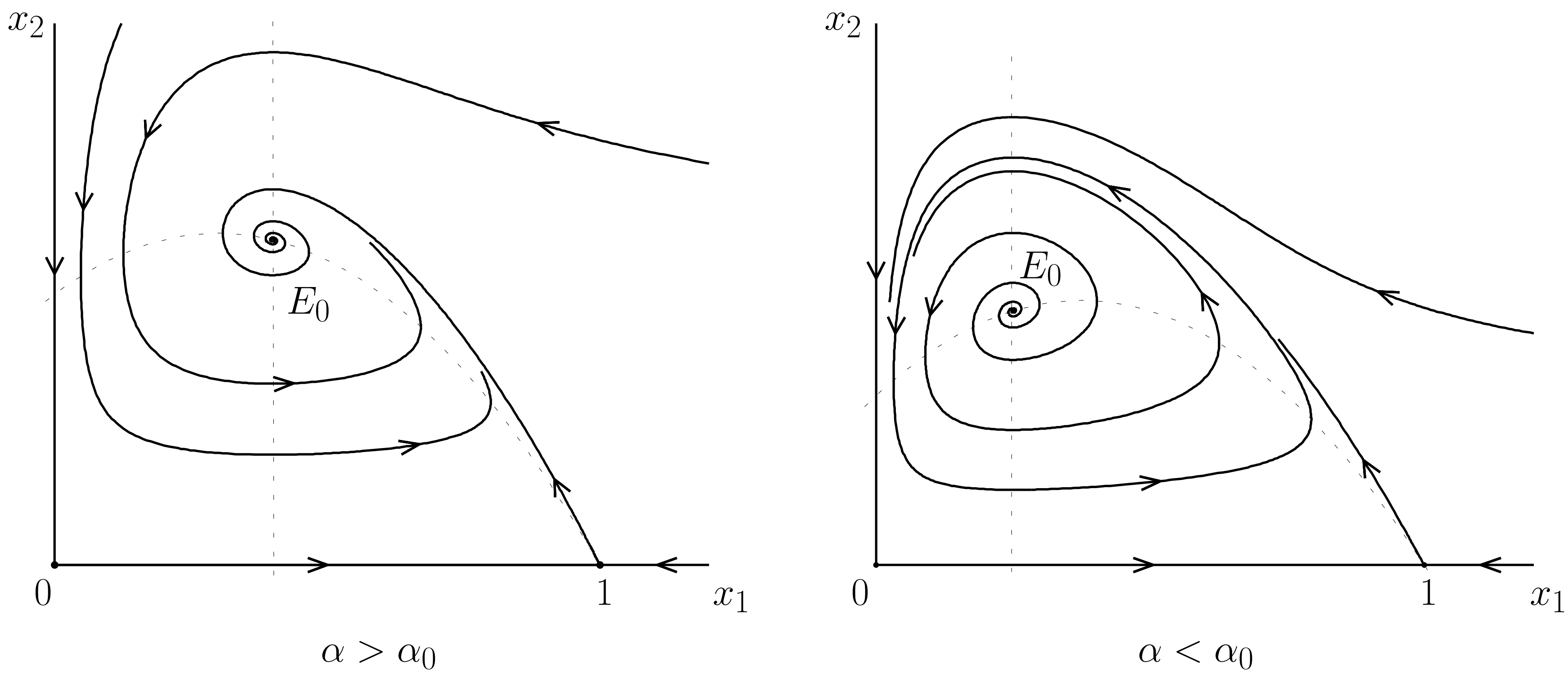
$$g_{20} = \langle p, B(q,q) \rangle = \frac{cd(c^2 - d^2 - rd) + i\omega c(c+d)^2}{(c+d)} \ ,$$

$$g_{11} = \langle p, B(q,\bar{q}) \rangle = -\frac{rcd^2}{(c+d)}, \quad g_{21} = \langle p, C(q,q,\bar{q}) \rangle = -3rc^2d^2,$$

and the first Lyapunov coefficient

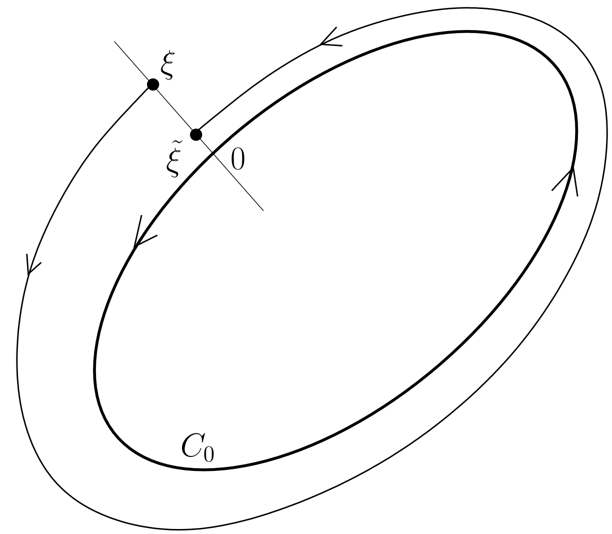
$$l_1(\alpha_0) = \frac{1}{2\omega^2}\ \mathrm{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rc^2d^2}{\omega} < 0.$$

Therefore, a **stable cycle** bifurcates from the equilibrium via the supercritical Hopf bifurcation for $\alpha < \alpha_0$.



One can prove that the cycle is **unique**.

3. LOCAL BIFURCATION OF CYCLES: $\mu = 1$



Parameter-dependent Poincaré map:

$$\xi \mapsto \tilde{\xi} = P(\xi, \alpha),$$

where $P(\xi, 0) = \xi + O(2) \quad (\mu = 1)$

Lemma 1 *If*

$$p_2(0) = \frac{1}{2} P_{\xi\xi}(0,0) \neq 0,$$

then there exists a smooth function $\delta = \delta(\alpha)$ such that the substitution $x = \xi + \delta(\alpha)$ reduces the map

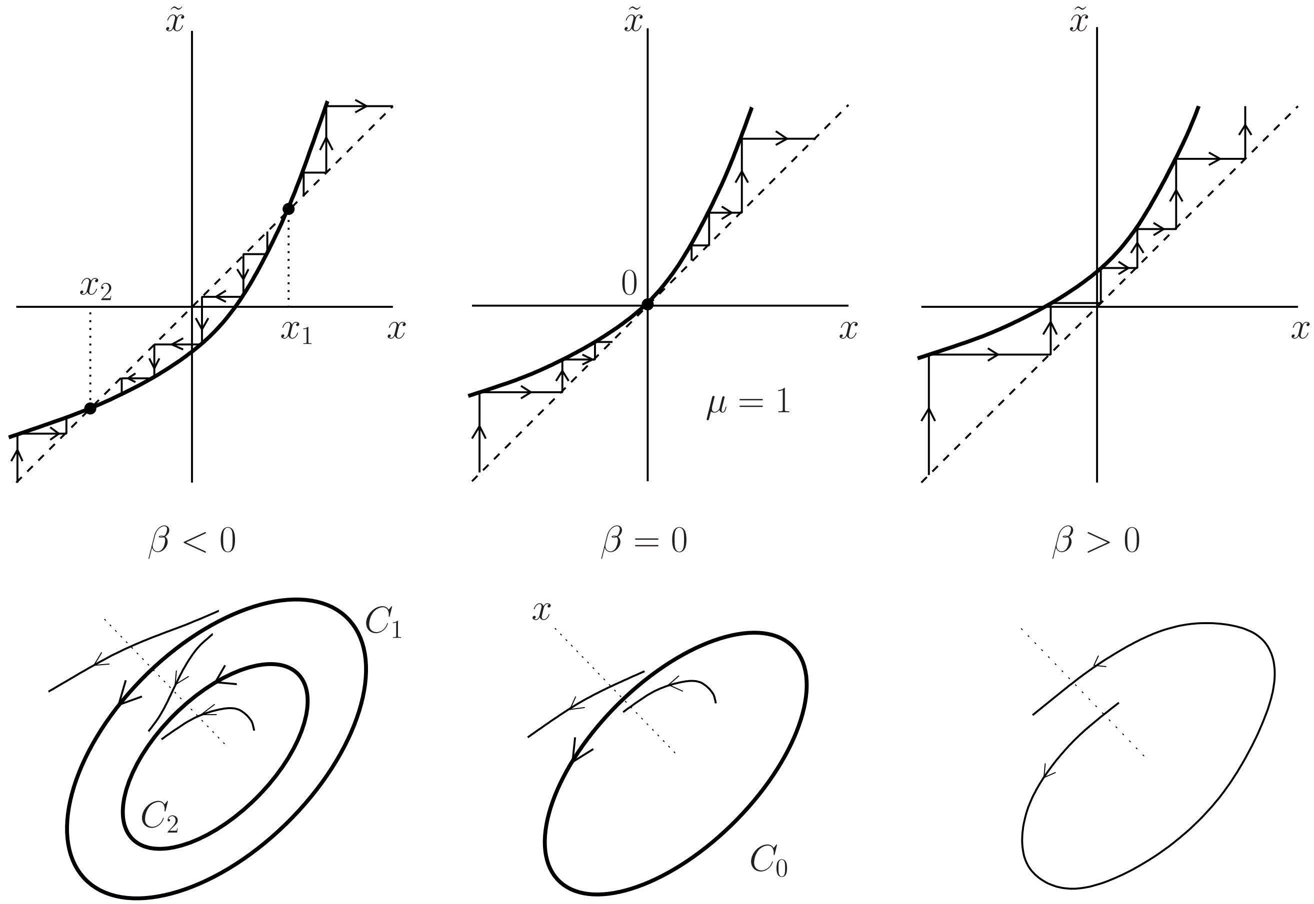
$$\xi \mapsto P(\xi, \alpha) = p_0(\alpha) + [1 + g(\alpha)]\xi + p_2(\alpha)\xi^2 + O(3),$$

where $g(0) = 0, p_0(0) = P(0,0) = 0$, to the form

$$x \mapsto \tilde{x} = \beta(\alpha) + x + b(\alpha)x^2 + O(3)$$

with $\beta(0) = 0$ and $b(0) = p_2(0) \neq 0$.

Cyclic fold: $x \mapsto \beta + x + bx^2,\ b > 0$



Two hyperbolic cycles (unstable C_1 and stable C_2) collide forming a non-hyperbolic cycle C_0, and disappear.

4. (GLOBAL) BIFURCATIONS OF CONNECTING ORBITS

- **Saddle homoclinic bifurcation**

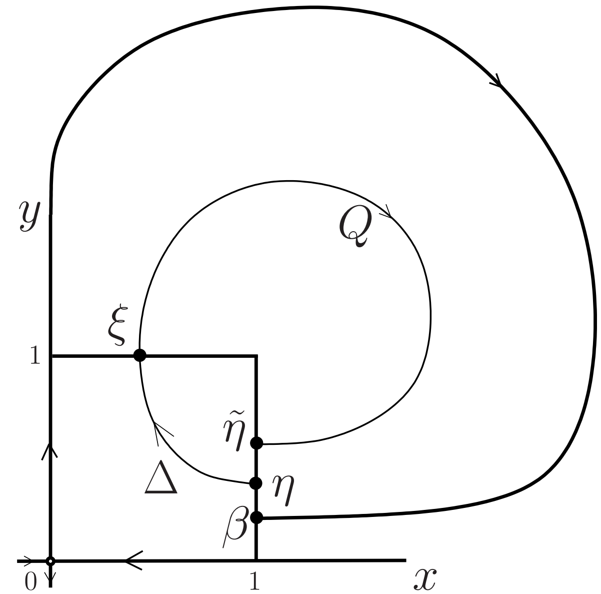
Singular map: $\eta \mapsto \xi = \eta^{-\frac{\lambda_1}{\lambda_2}}$.

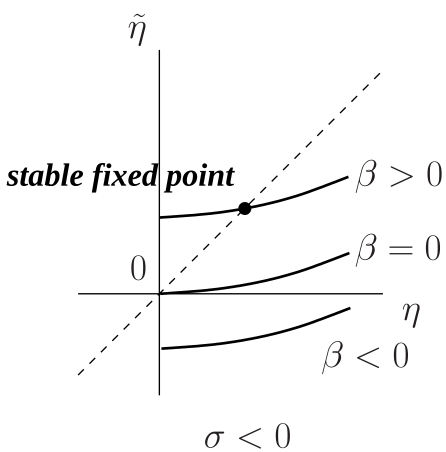
Regular map:

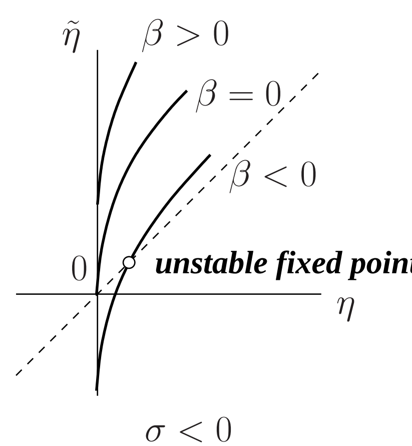
$$\xi \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\xi + O(2), \quad A(0) > 0.$$

Poincaré map:

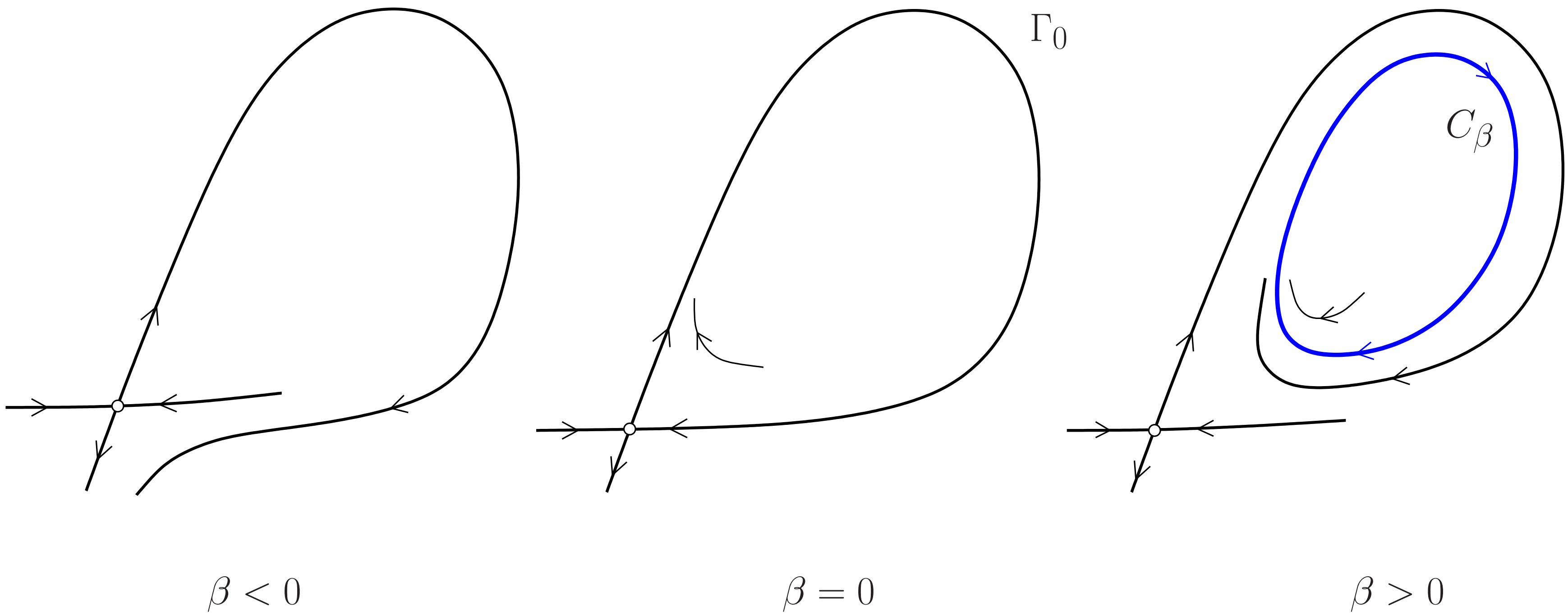
$$\eta \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\eta^{-\frac{\lambda_1}{\lambda_2}} + \ldots$$





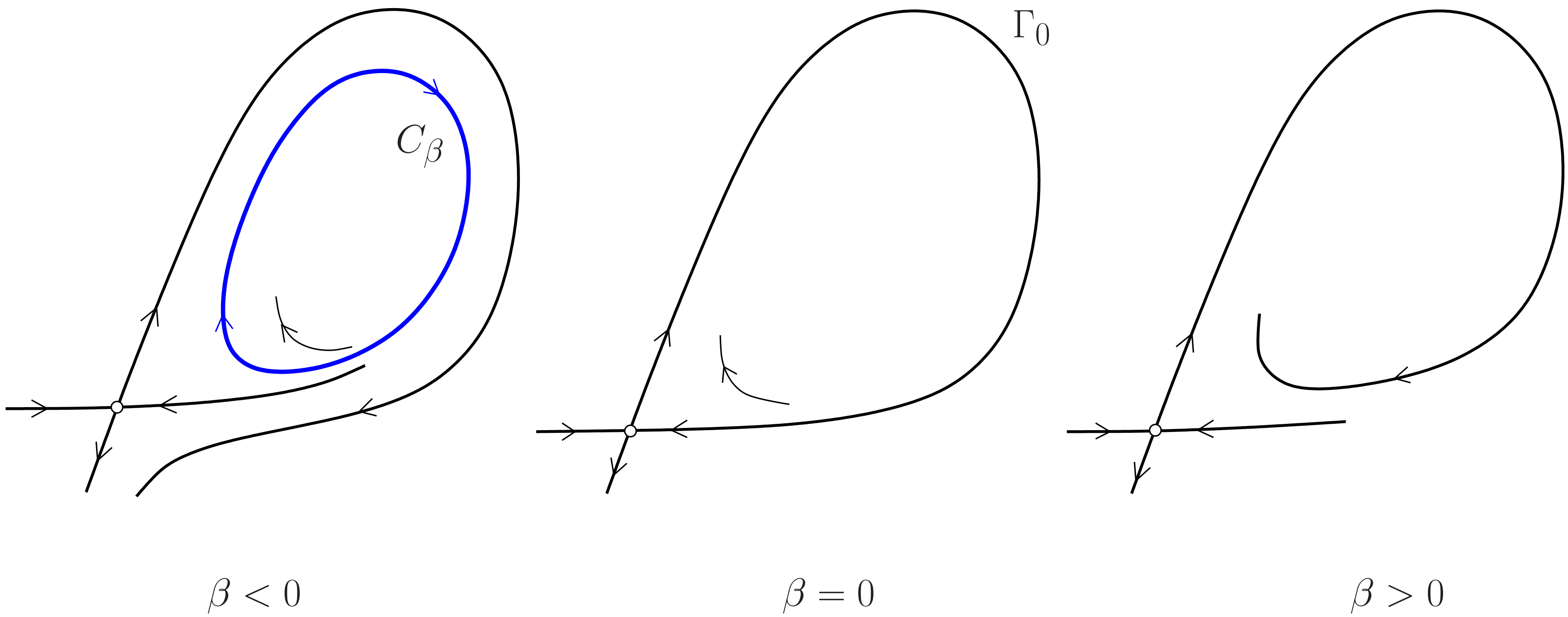


Saddle homoclinic bifurcation: $\sigma < 0$



A stable cycle C_β bifurcates from Γ_0 while the separatrices exchange.

Saddle homoclinic bifurcation: $\sigma > 0$

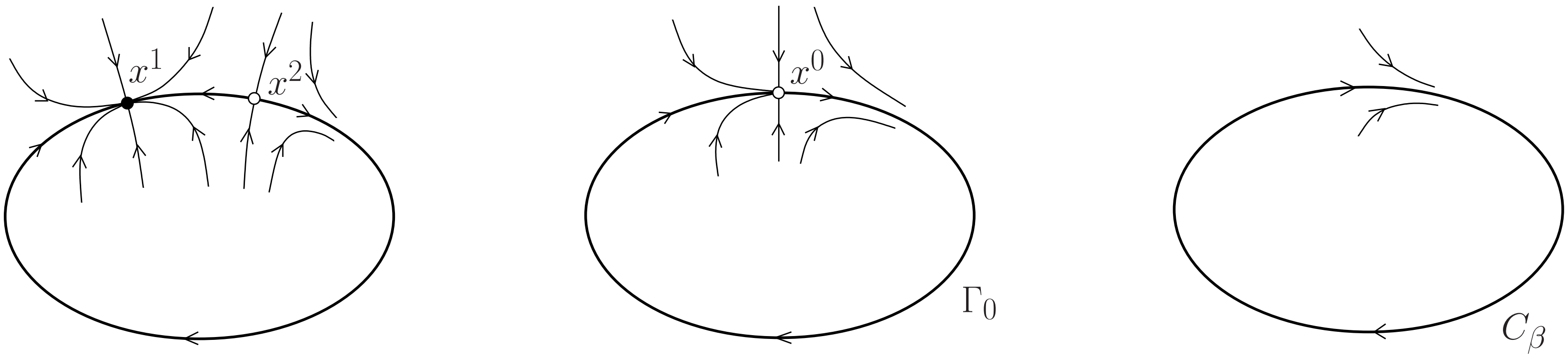


An unstable cycle C_β bifurcates from Γ_0 while the separatrices exchange.

Theorem 5 (Melnikov)

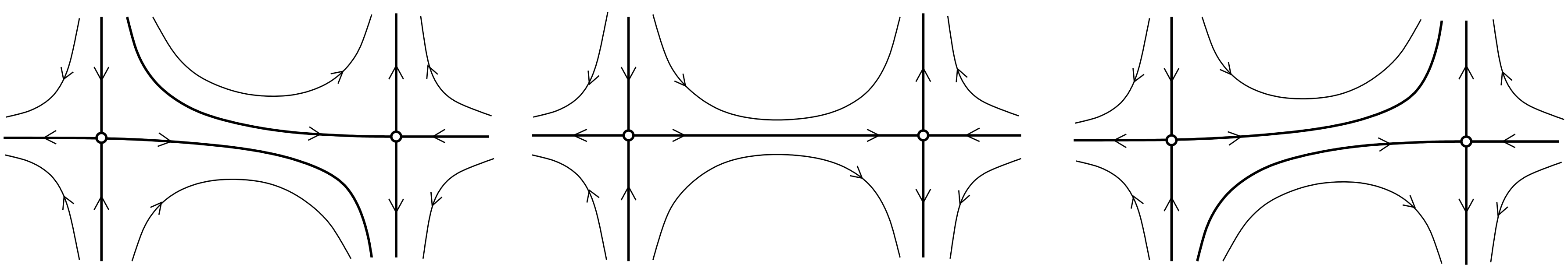
$$\beta'(0) \neq 0 \;\Leftrightarrow \int_{-\infty}^{\infty} \exp\left(-\int_0^t \mathrm{div}\; f(X^0(s))ds\right)\left(f_1\frac{\partial f_2}{\partial \alpha} - f_2\frac{\partial f_1}{\partial \alpha}\right)(X^0(t))dt \neq 0$$

- **Homoclinic saddle-node bifurcation:**



$\beta < 0$ $\beta = 0$ $\beta > 0$

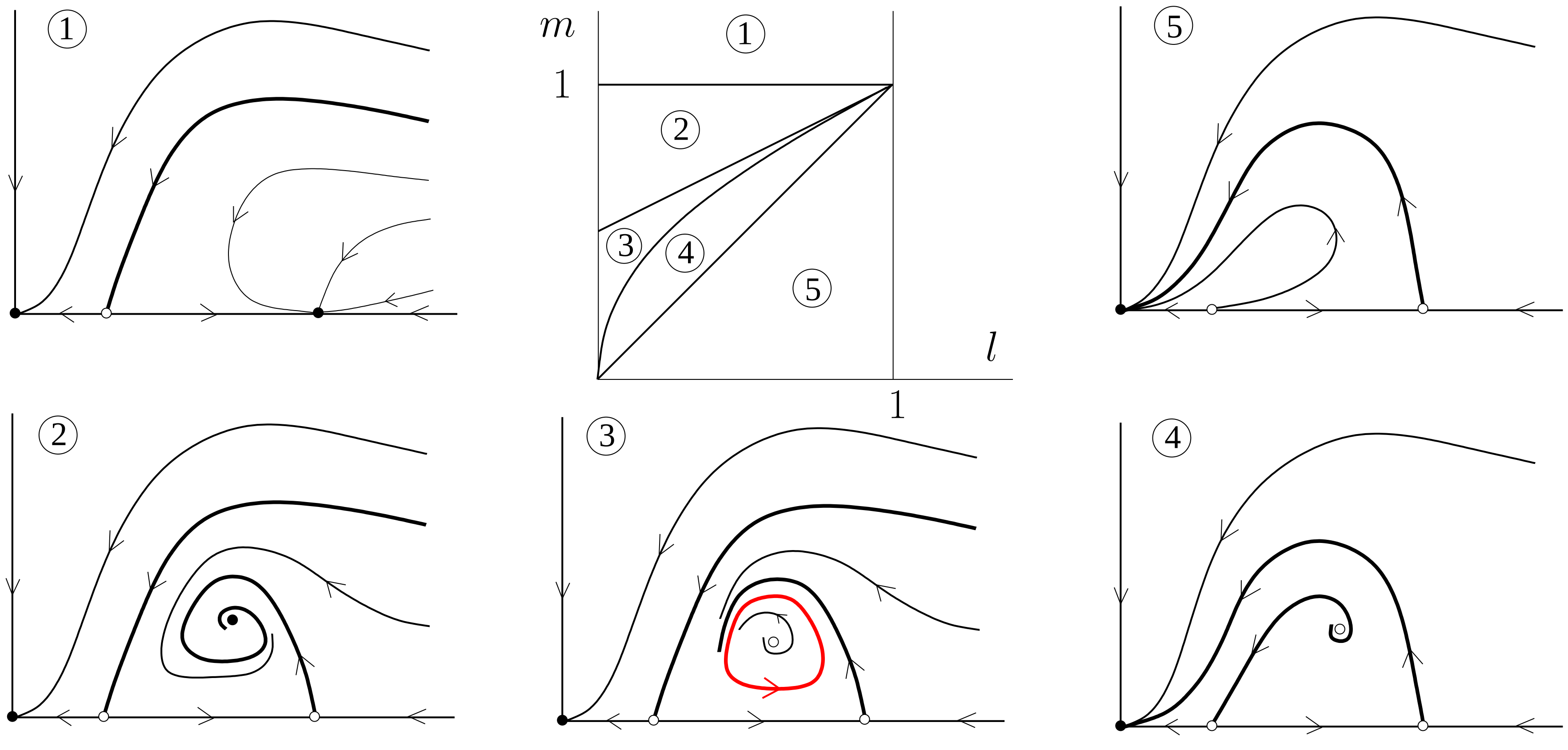
- **Heteroclinic saddle bifurcation:**



$\beta < 0$ $\beta = 0$ $\beta < 0$

Example: Allee effect in a prey-predator system

$$\begin{cases} \dot{x} & = & x(x-l)(1-x) - xy, \\ \dot{y} & = & -\gamma y(m-x). \end{cases}$$



Remarks:

1. There are **no** other codim 1 bifurcations in generic smooth 2D ODEs.

2. Heteroclinic bifurcation points can **accumulate**: