Bogdanov-Takens bifurcations: An interplay between symbolic and numerical analysis

Yuri A. Kuznetsov

April 16, 2014

Universiteit Utrecht

UNIVERSITY OF TWENTE.

References

- Yu.A. Kuznetsov
  Practical computation of normal forms on center manifolds at degenerate Bogdanov-Takens bifurcations
  Int. J. Bifurcation & Chaos: **15** (2005), 3535-3546
- S.M. Baer, B.W. Kooi, Yu.A. Kuznetsov, and H.R. Thieme
  Multiparametric bifurcation analysis of a basic two-stage population model
  SIAM J. Appl. Math.: **66** (2006), 1339-1365
- Yu.A. Kuznetsov, H.G.E. Meijer, B. Al Hdaibat, and W. Govaerts
  Improved homoclinic predictor for Bogdanov-Takens bifurcation
  Int. J. Bifurcation & Chaos: **24** (2014) [in press]

THE SPEAKER IS THANKFUL TO JAN SANDERS FOR DISCUSSIONS RELATED AND UNRELATED TO THIS WORK

Contents

Critical normal form

- Consider a generic smooth family of planar autonomous ODEs

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R}^m$$

- Suppose that $f(0,0) = 0$ and $A = f_x(0,0)$ has one double zero eigenvalue with the Jordan block

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

  This indicates a *Bogdanov-Takens* (*BT*) *bifurcation.*

- The ODE at the BT-bifurcation is formally smoothly equivalent to

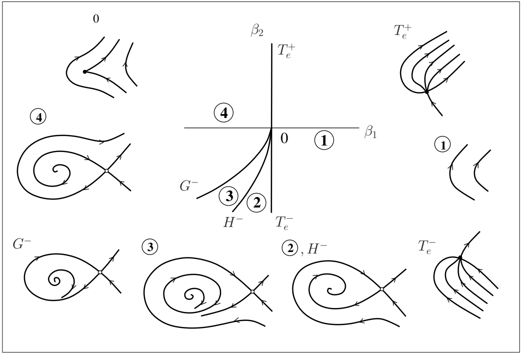
$$\begin{cases} \dot{w}_0 &= w_1 \\ \dot{w}_1 &= \sum_{k\geq 2} \left(a_k w_0^k + b_k w_0^{k-1} w_1 \right) \end{cases}$$

Classical codim 2 BT bifurcation

- Versal unfolding when $a_2 b_2 \neq 0$ (Bogdanov[1975], Takens[1974]):

$$\begin{cases} \dot{w}_0 = w_1 \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + a_2 w_0^2 + b_2 w_0 w_1 \end{cases}$$

- The bifurcation diagram:



Approximation of the homoclinic solution

The singular rescaling

$$w_0 = \frac{\varepsilon^2}{a}u, \quad w_1 = \frac{\varepsilon^3}{a}v$$
$$\beta_1 = -\frac{4}{a}\varepsilon^4, \quad \beta_2 = \frac{b}{a}\varepsilon^2\tau, \quad \varepsilon t = s$$

in the versal unfolding

$$\begin{cases} \dot{w}_0 = w_1 \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \end{cases}$$

gives the perturbed Hamiltonian system

$$\begin{cases} \dot{u} = v \\ \dot{v} = -4 + u^2 + \varepsilon \frac{b}{a} v(\tau + u) \end{cases}$$

- Trivial branch $(u_0, v_0, 0, \tau)$ of homoclinic solutions with $v(0) = 0$:

$$\begin{pmatrix} u_0(s) \\ v_0(s) \end{pmatrix} = 2 \begin{pmatrix} 1 - 3\text{sech}^2(s) \\ 6\text{sech}^2(s)\tanh(s) \end{pmatrix}.$$

- Bifurcation point $\tau_0 = \dfrac{10}{7}$
- Nontrivial branch of homoclinic solutions with $v(0) = 0$:

$$\begin{pmatrix} u \\ v \\ \varepsilon \\ \tau \end{pmatrix} = \sum_{l=0}^{L} \varepsilon^l \begin{pmatrix} u_l \\ v_l \\ 0 \\ \tau_l \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

where L is the order of approximation.

Linear inhomogeneous systems

- ε^1-terms:

$$\begin{cases} \dot{u}_1 = v_1 \\ \dot{v}_1 = 2u_0u_1 + \frac{b}{a}v_0(\tau_0 + u_0) \end{cases}$$

- ε^2-terms:

$$\begin{cases} \dot{u}_2 = v_2 \\ \dot{v}_2 = 2u_0u_2 + \frac{b}{a}v_0(\tau_1 + u_1) + \frac{b}{a}v_1(\tau_0 + u_0) + u_1^2 \end{cases}$$

- ε^3-terms:

$$\begin{cases} \dot{u}_3 = v_3 \\ \dot{v}_3 = 2u_0u_3 + \frac{b}{a}v_0(\tau_2 + u_2) + \frac{b}{a}v_1(\tau_1 + u_1) \\ \qquad + \frac{b}{a}v_2(\tau_0 + u_0) + 2u_1u_2 \end{cases}$$

- Bounded solutions (u_j, v_j) exist iff

$$\tau_0 = \frac{10}{7}, \;\; \tau_1 = 0, \;\; \tau_2 = \frac{288 b^2}{2401 a^2}$$

- The solutions

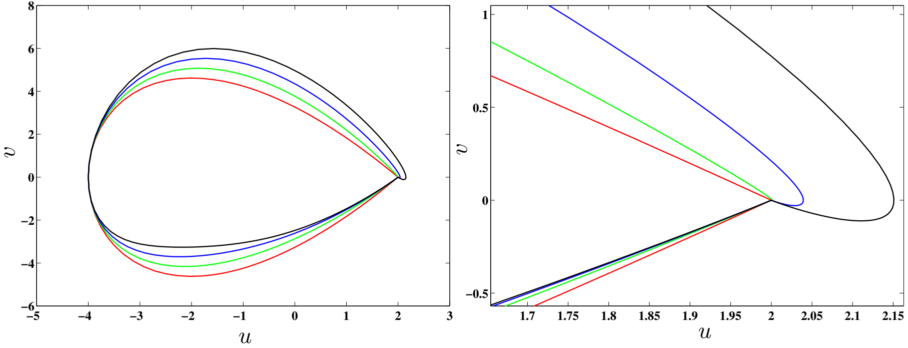
$$u_1(s) = -\frac{72b}{7a}\frac{\sinh(s)\log(\cosh(s))}{\cosh^3(s)}$$

$$v_1(s) = -\frac{72b}{7a}\frac{\sinh^2(s)+(1-2\sinh^2(s))\log(\cosh(s))}{\cosh^4(s)}$$

- Tangent approximation of the homoclinic branch:

$$(u, v, \varepsilon, \tau) = (u_0 + \varepsilon u_1, v_0 + \varepsilon v_1, \varepsilon, \tau_0)$$

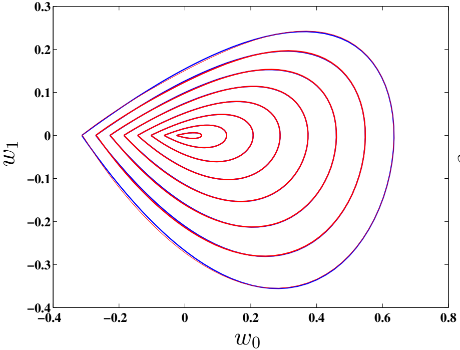
Tangent predictor

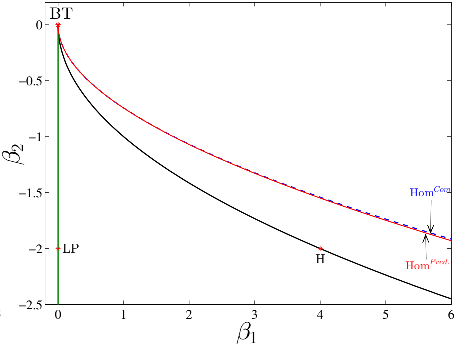


$\varepsilon = 0.0, 0.2, 0.4, 0.6$

$$
\begin{aligned}
u_2 &= -\frac{216b^2}{49a^2}\frac{\log^2(\cosh(t))(\cosh(2t)-2)}{\cosh^4(t)} \\
&- \frac{216b^2}{49a^2}\frac{\log(\cosh(t))(1-\cosh(2t))}{\cosh^4(t)} \\
&- \frac{18b^2}{49a^2}\frac{(6t\sinh(2t)-7\cosh(2t)+8)}{\cosh^4(t)} \\
v_2 &= \frac{216b^2}{49a^2}\frac{t(2\cosh^2(t)-3)}{\cosh^4(t)} \\
&+ \frac{288b^2}{49a^2}\frac{\sinh(t)(3\log^2(\cosh(t))-6\log(\cosh(t)))}{\cosh^3(t)} \\
&- \frac{216b^2}{49a^2}\frac{\sinh(t)(12\log^2(\cosh(t))-14\log(\cosh(t)))}{\cosh^5(t)} \\
&- \frac{288b^2}{49a^2}\frac{\sinh(t)}{\cosh^3(t)} + \frac{648b^2}{49a^2}\frac{\sinh(t)}{\cosh^5(t)}
\end{aligned}
$$

Second-order predictor





$$\begin{cases} \dot{w}_0 = w_1 \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 - w_0^2 + w_0 w_1 \end{cases}$$

Codim 3 BT bifurcation with double equilibrium

- If $b_2 = 0$ but $a_2 \neq 0$, the critical ODE is smoothly orbitally equivalent to

$$\begin{cases} \dot{w}_0 &= w_1 \\ \dot{w}_1 &= a_2 w_0^2 + b_4 w_0^3 w_1 + O(\|(w_0, w_1)\|^5) \end{cases}$$

- Versal unfolding when $b_2 = 0$ but $a_2 b_4 \neq 0$ (Berezovskaya & Khibnik [1985], Dumortier, Roussarie & Sotomayor [1987]):

$$\begin{cases} \dot{w}_0 &= w_1 \\ \dot{w}_1 &= \beta_1 + \beta_2 w_1 + \beta_3 w_0 w_1 + a_2 w_0^2 + b_4 w_0^3 w_1 \end{cases}$$

- The bifurcation diagram includes a neutral saddle homoclinic and a degenerate Andronov-Hopf (Bautin) bifurcation curves.

Codim 3 BT bifurcation with triple equilibrium $(b_2 > 0)$

- If $a_2 = 0$ but $b_2 a_3 \neq 0$, the critical ODE is smoothly orbitally equivalent to
$$\begin{cases} \dot{w}_0 = w_1 \\ \dot{w}_1 = a_3 w_0^3 + b_2 w_0 w_1 + b_3' w_0^2 w_1 + O(\|(w_0, w_1)\|^5) \end{cases}$$
where $b_3' = b_3 - \dfrac{3 b_2 a_4}{5 a_3}$.
- If $a_3 > 0$ the origin is a topological *saddle*. If $a_3 < 0$, $b_2^2 + 8a_3 < 0$ and $b_3' \neq 0$, the origin is a topological *focus*. If $a_3 < 0$ and $b_2^2 + 8a_3 > 0$, the origin has one *elliptic sector*.
- "Versal" unfolding in all cases (Dumortier, Roussarie, Sotomayor & Żołądek [1991]):

$$\begin{cases} \dot{w}_0 = w_1 \\ \dot{w}_1 = \beta_1 + \beta_2 w_0 + \beta_3 w_1 + a_3 w_0^3 + b_2 w_0 w_1 + b_3' w_0^2 w_1 \end{cases}$$

Bifurcations of a triple equilibrium with elliptic sector

- Truncated and rescaled critical normal form:

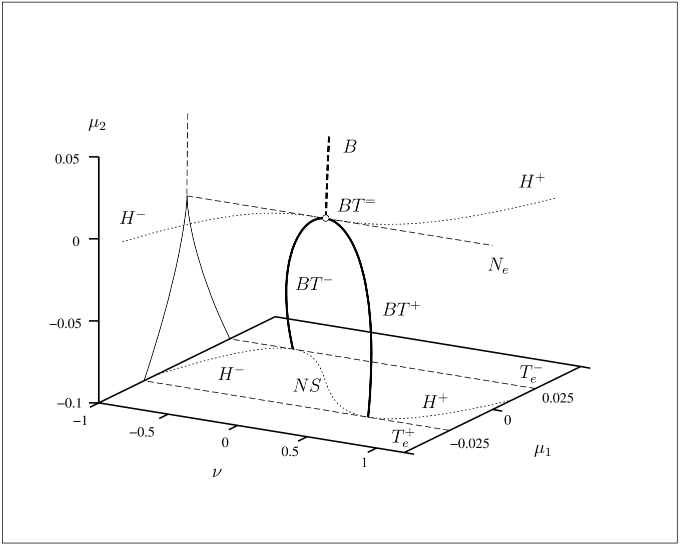
$$\begin{cases} \dot{\xi} & = & \eta \\ \dot{\eta} & = & \beta\xi\eta + \epsilon_1\xi^3 + \epsilon_2\xi^2\eta \end{cases}$$

  where $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$, and $\beta > 0$.

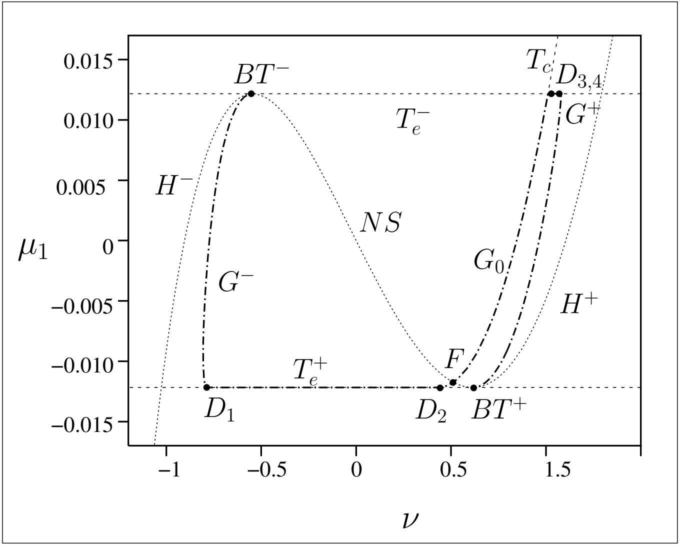
- Saddle case: $\epsilon_1 = 1$, any ϵ_2 and β;
  Focus case: $\epsilon_1 = -1$ and $0 < \beta < 2\sqrt{2}$;
  Elliptic case: $\epsilon_1 = -1$ and $2\sqrt{2} < \beta$.
- Unfolding:

$$\begin{cases} \dot{\xi} & = & \eta \\ \dot{\eta} & = & -\mu_1 - \mu_2\xi + \nu\eta + \beta\xi\eta - \xi^3 - \xi^2\eta \end{cases}$$

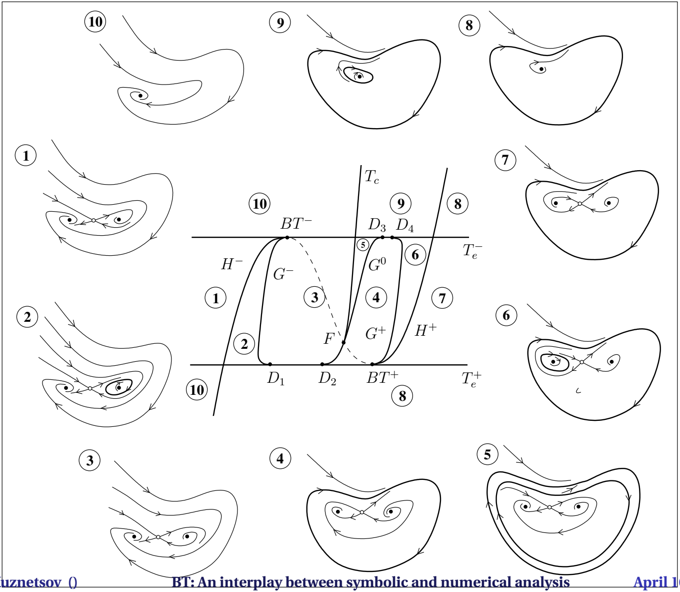
Local bifurcations: $\beta = 3.175849820$



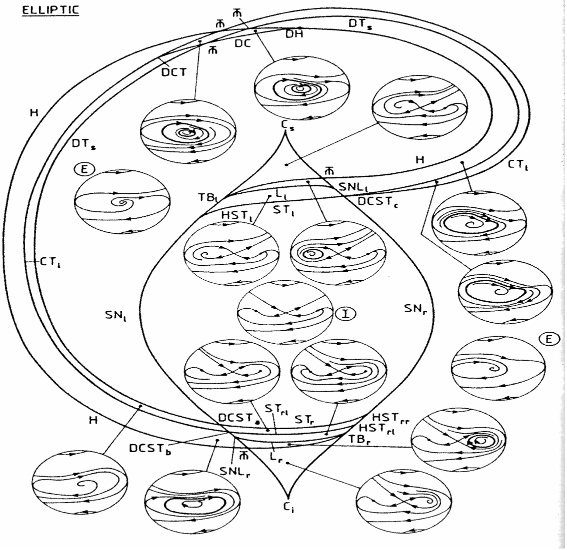
Local and global bifurcations: $\mu_2 = 0.1, \beta = 3.175849820$



Schematic bifurcation diagram in the elliptic case



Theoretical bifurcation diagram [Dumortier et al. 1991]



Elliptic versus focus case

- The numerical bifurcation diagram differs drastically from the theoretical bifurcation diagram for the elliptic case given by Dumortier et al. [1991] who studied phase portraits in a *fixed* small neighborhood of the origin.
- It turns out that generic two-parameter slices in the elliptic case are topologically equivalent to those in the focus case.
- However, the inner limit cycle demonstrates rapid amplitude changes ("canard-like" behavior) near the bifurcation curve T_c.
- The "big" homoclinic orbit to the neutral saddle (point F) shrinks not to the origin of the phase plane, but to the boundary of the elliptic sector that has a finite size in the unfolding.

Combined reduction/normalization technique

- Critical ODE: $\dot{x} = F(x), \ \ x \in \mathbb{R}^n$,
  with Taylor expansion

$$F(x) = Ax + \tfrac{1}{2}B(x,x) + \tfrac{1}{6}C(x,x,x) + \tfrac{1}{24}D(x,x,x,x) + O(\|x\|^5).$$

- Eigenvectors: $q_{0,1}, p_{0,1} \in \mathbb{R}^n$,

$$Aq_0 = 0,\ Aq_1 = q_0,\ A^T p_1 = 0,\ A^T p_0 = p_1$$

  with $\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1, \langle p_0, q_1 \rangle = \langle p_1, q_0 \rangle = 0.$
- Critical center manifold:

$$x = H(w_0, w_1) = w_0 q_0 + w_1 q_1 + \sum_{2 \le j+k \le 4} \frac{1}{j!k!} h_{jk} w_0^j w_1^k + O(\|(w_0, w_1)\|^5$$

  where $(w_0, w_1) \in \mathbb{R}^2, h_{jk} \in \mathbb{R}^n$.

- Critical normal form:

$$\begin{cases} \dot{w}_0 &=& w_1 \\ \dot{w}_1 &=& a_2 w_0^2 + b_2 w_0 w_1 + a_3 w_0^3 + b_3 w_0^2 w_1 + a_4 w_0^4 + b_4 w_0^3 w_1 \\ &+& O(\|(w_0, w_1)\|^5) \end{cases}$$

- Homological equation:

$$H_{w_0} \dot{w}_0 + H_{w_1} \dot{w}_1 = F(H(w_0, w_1))$$

- Collecting the $w_0^j w_1^k$-terms give singular linear systems for h_{jk}. Since these systems must be solvable, their right-hand sides should be orthogonal to p_1. Some of these Fredholm conditions will define the normal form coefficients, others can be satisfied using a freedom in selecting solutions of singular linear systems appearing at lower-order terms.

Quadratic terms

- The w_0^2-terms give

$$Ah_{20} = 2a_2 q_1 - B(q_0, q_0)$$

The Fredholm solvability condition for this system implies

$$a_2 = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle$$

- The $w_0 w_1$-terms give

$$Ah_{11} = b_2 q_1 + h_{20} - B(q_0, q_1)$$

Its solvability leads to the expression

$$b_2 = \langle p_1, B(q_0, q_1)\rangle - \langle p_1, h_{20}\rangle$$

- The w_1^2-terms give

$$Ah_{02} = 2h_{11} - B(q_1, q_1)$$

Since

$$\langle p_1, h_{11} \rangle = \langle p_0, h_{20} \rangle - \langle p_0, B(q_0, q_1) \rangle$$

we get

$$\langle p_1, 2h_{11} - B(q_1, q_1) \rangle = 2\langle p_0, h_{20} \rangle - 2\langle p_0, B(q_0, q_1) \rangle - \langle p_1, B(q_1, q_1) \rangle$$

The substitution $h_{20} \mapsto h_{20} + \delta_0 q_0$ with a properly selected δ_0 makes the right-hand side of this equation equal to zero. This does not affect the coefficient b_2, because $\langle p_1, q_0 \rangle = 0$.

Cubic terms

- The w_0^3-terms give

$$Ah_{30} = 6q_1 a_3 + 6h_{11} a_2 - 3B(h_{20}, q_0) - C(q_0, q_0, q_0)$$

Its solvability implies

$$a_3 = \frac{1}{6}\langle p_1, C(q_0, q_0, q_0)\rangle + \frac{1}{2}\langle p_1, B(h_{20}, q_0)\rangle - a_2\langle p_1, h_{11}\rangle$$

- The $w_0^2 w_1$-terms give

$$Ah_{21} = h_{30} + 2b_3 q_1 + 2a_2 h_{02} + 2b_2 h_{11} - 2B(h_{11}, q_0) - B(h_{20}, q_1) - C(q_0, q_0, q_1)$$

which solvability implies

$$\begin{aligned} b_3 &= \frac{1}{2}\langle p_1, C(q_0, q_0, q_1) + 2B(h_{11}, q_0) + B(h_{20}, q_1)\rangle \\ &- \frac{1}{2}\langle p_1, h_{30} + 2a_2 h_{02} + 2b_2 h_{11}\rangle \end{aligned}$$

- The singular linear systems resulting from the $w_0 w_1^2$- and w_1^3-terms,

$$Ah_{12} = 2h_{21} + 2b_2 h_{02} - B(h_{02}, q_0) - 2B(h_{11}, q_1) - C(q_0, q_1, q_1)$$

and

$$Ah_{03} = 3h_{12} - 3B(h_{02}, q_1) - C(q_1, q_1, q_1)$$

can be made solvable for any h_{02} by substituting $h_{30} \mapsto h_{30} + \delta_1 q_0$ and then $h_{21} \mapsto h_{21} + \delta_2 q_0$ with properly selected δ_1 and δ_2. This does not change b_3.

Fourth-order terms

- The w_0^4-terms imply

$$\begin{aligned} a_4 &= \tfrac{1}{24}\langle p_1, D(q_0,q_0,q_0,q_0) + 6C(h_{20},q_0,q_0)\rangle \\ &+ \tfrac{1}{24}\langle p_1, 4B(h_{30},q_0) + 3B(h_{20},h_{20})\rangle \\ &- \tfrac{1}{2}a_2\langle p_1, h_{21}\rangle - a_3\langle p_1, h_{11}\rangle \end{aligned}$$

- The $w_0^3 w_1$-terms imply

$$\begin{aligned} b_4 &= \tfrac{1}{6}\langle p_1, D(q_0,q_0,q_0,q_1) + 3C(h_{20},q_0,q_1) + 3C(h_{11},q_0,q_0)\rangle \\ &+ \tfrac{1}{6}\langle p_1, 3B(h_{21},q_0) + 3B(h_{11},h_{20}) + B(h_{30},q_1)\rangle \\ &- \tfrac{1}{6}\langle p_1, h_{40}\rangle - \tfrac{1}{2}b_2\langle p_1, h_{21}\rangle \\ &- \langle p_1, a_2 h_{12} + a_3 h_{02} + b_3 h_{11}\rangle \end{aligned}$$

Some simplifications

- Since $\langle p_1, h_{20}\rangle = -\langle p_0, B(q_0,q_0)\rangle$, we obtain

$$b_2 = \langle p_0, B(q_0,q_0)\rangle + \langle p_1, B(q_0,q_1)\rangle$$

- Since $\langle p_1, h_{11}\rangle = \frac{1}{2}\langle p_1, B(q_1,q_1)\rangle$, we obtain

$$a_3 = \frac{1}{6}\langle p_1, C(q_0,q_0,q_0)\rangle + \frac{1}{2}\langle p_1, B(h_{20},q_0)\rangle - \frac{1}{2}a_2\langle p_1, B(q_1,q_1)\rangle$$

- Similarly, we obtain

$$\begin{array}{rcl} b_3 & = & \frac{1}{2}\langle p_1, C(q_0,q_0,q_1) + 2B(h_{11},q_0) + B(h_{20},q_1)\rangle \\ & + & \frac{1}{2}\langle p_0, C(q_0,q_0,q_0) + 3B(h_{20},q_0)\rangle \\ & - & \frac{1}{2}b_2\langle p_1, B(q_1,q_1)\rangle + a_2\langle p_0, B(q_1,q_1)\rangle \\ & - & 5a_2\langle p_0, h_{11}\rangle \end{array}$$

Parameter-dependent center manifold reduction at codim 2 BT

- The ODE system:

$$\begin{aligned}\dot{x} = f(x,\alpha) &= Ax + \tfrac{1}{2}B(x,x)\\ &+ J_1\alpha + A_1(x,\alpha) + \tfrac{1}{2}J_2(\alpha,\alpha)\\ &+ \mathcal{O}(\|x\|^3 + \|x\|\|\alpha\|^2 + \|x\|^2\|\alpha\| + \|\alpha\|^3)\end{aligned}$$

- The normal form:

$$\begin{aligned}\dot{w} = G(w,\beta) &= \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \end{pmatrix}\\ &+ \mathcal{O}(\|w\|^3 + \|\beta\|\|w\|^2)\end{aligned}$$

where $G : \mathbb{R}^{n_c+2} \to \mathbb{R}^{n_c}$.

- The center manifold:

$$\begin{aligned} x = H(w,\beta) &= H_{01}\beta + [q_0, q_1]w + \tfrac{1}{2}H_{20,0}w_0^2 \\ &+ H_{20,1}w_0w_1 + \tfrac{1}{2}H_{02,2}\beta_2^2 \\ &+ H_{21,0}\beta_1 w_0 + H_{21,1}\beta_1 w_1 \\ &+ H_{12,0}\beta_2 w_0 + H_{12,1}\beta_2 w_1 \\ &+ \mathcal{O}(|w_0|^3 + |w_0^2 w_1| + w_1^2) + \mathcal{O}(\beta_1^2 + |\beta_1\beta_2| + |\beta_2|^3) \end{aligned}$$

- The parameter transformation

$$\alpha = K(\beta) = K_1\beta + \tfrac{1}{2}K_2\beta_2^2 + \mathcal{O}(\beta_1^2 + |\beta_1\beta_2| + |\beta_2|^3)$$

- The homological equation:

$$H_w(w,\beta)G(w,\beta) = f(H(w,\beta), K(\beta))$$

Quadratic w-terms give:

- The solvability for $H_{20,0}$ and $H_{20,1}$ implies

$$\begin{aligned} a &= \tfrac{1}{2} p_1^T B(q_0, q_0) \\ b &= p_0^T B(q_0, q_0) + p_1^T B(q_0, q_1) \end{aligned}$$

- With such a and b,

$$\begin{aligned} H_{20,0} &= A^{INV}(2aq_1 - B(q_0, q_0)) \\ H_{20,1} &= A^{INV}(bq_1 + H_{20,0} - B(q_0, q_1)) \end{aligned}$$

where $x = A^{INV} y$ is defined by solving the non-singular bordered system

$$\begin{pmatrix} A & p_1 \\ q_0^T & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

The β- and βw-terms

For $K_1 = [K_{1,0}, K_{1,1}]$ and $H_{01} = [H_{01,0}, H_{01,1}]$ we get the equations

$$
\begin{aligned}
AH_{01} + J_1 K_1 &= [q_1, 0] \\
p_1^T B(q_0, H_{01}) + p_1^T A_1(q_0, K_1) &= \frac{1}{2}\left[p_1^T B(q_1, q_1), 0\right] \\
p_0^T B(q_0, H_{01}) + p_1^T B(q_1, H_{01}) &+ p_0^T A_1(q_0, K_1) + p_1^T A_1(q_1, K_1) \\
&= [-p_0^T B(q_1, q_1) + 3 p_0^T H_{20,1}, 1]
\end{aligned}
$$

that is the linear system

$$
\begin{pmatrix} A & J_1 \\ p_1^T \mathrm{B} q_0 & p_1^T A_1 q_0 \\ p_0^T \mathrm{B} q_0 + p_1^T \mathrm{B} q_1 & p_0^T A_1 q_0 + p_1^T A_1 q_1 \end{pmatrix} \begin{pmatrix} H_{01} \\ K_1 \end{pmatrix}
= \begin{pmatrix} q_1 & 0 \\ \frac{1}{2} p_1^T B(q_1, q_1) & 0 \\ -p_0^T B(q_1, q_1) + 3 p_0^T H_{20,1} & 1 \end{pmatrix}
$$

$$\begin{aligned} K_2 &= -(p_1^T z) K_{1,0} \\ H_{02,2} &= -A^{INV}(z + J_1 K_2) \end{aligned}$$

where

$$\begin{aligned} z &= B(H_{01,1}, H_{01,1}) + 2A_1(H_{01,1}, K_{1,1}) \\ &+ J_2(K_{1,1}, K_{1,1}) \end{aligned}$$

as well as

$$H_{12,0} = -A^{INV}(B(q_0, H_{01,1}) + A_1(q_0, K_{1,1}))$$

The second-order homoclinic predictor

$$\alpha = \varepsilon^2 \frac{10b}{7a} K_{1,1} + \varepsilon^4 \left(-\frac{4}{a} K_{1,0} + \frac{50b^2}{49a^2} K_2 + \frac{288b^3}{2401a^3} K_{1,1} \right) + \mathcal{O}(\varepsilon^5)$$

and

$$\begin{aligned} x(t) = {} & \varepsilon^2 \left(\frac{10b}{7a} H_{01,1} + \frac{1}{a} u_0(\varepsilon t) q_0 \right) \\ & + \varepsilon^3 \left(\frac{1}{a} v_0(\varepsilon t) q_1 + \frac{1}{a} u_1(\varepsilon t) q_0 \right) \\ & + \varepsilon^4 \left(-\frac{4}{a} H_{01,0} + \frac{50b^2}{49a^2} H_{02,2} + \frac{288b^3}{2401a^3} H_{01,1} \right. \\ & \qquad + \frac{1}{a} u_2(\varepsilon t) q_0 + \frac{1}{a} v_1(\varepsilon t) q_1 \\ & \qquad \left. + \frac{1}{2a^2} H_{20,0} u_0(\varepsilon t)^2 + \frac{10b}{7a^2} H_{12,0} u_0(\varepsilon t) \right) \\ & + \mathcal{O}(\varepsilon^5) \end{aligned}$$

Implications for numerical bifurcation software

- Symbolic vs. numeric.
- Actual implementation in CONTENT and MATCONT.
- Predictors for homoclinic bifurcations at ZH and HH.