Numerical bifurcation analysis of dynamical systems:

Recent progress and perspectives

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1. Mathematical analysis of deterministic systems
2. Equilibria of ODEs and their bifurcations
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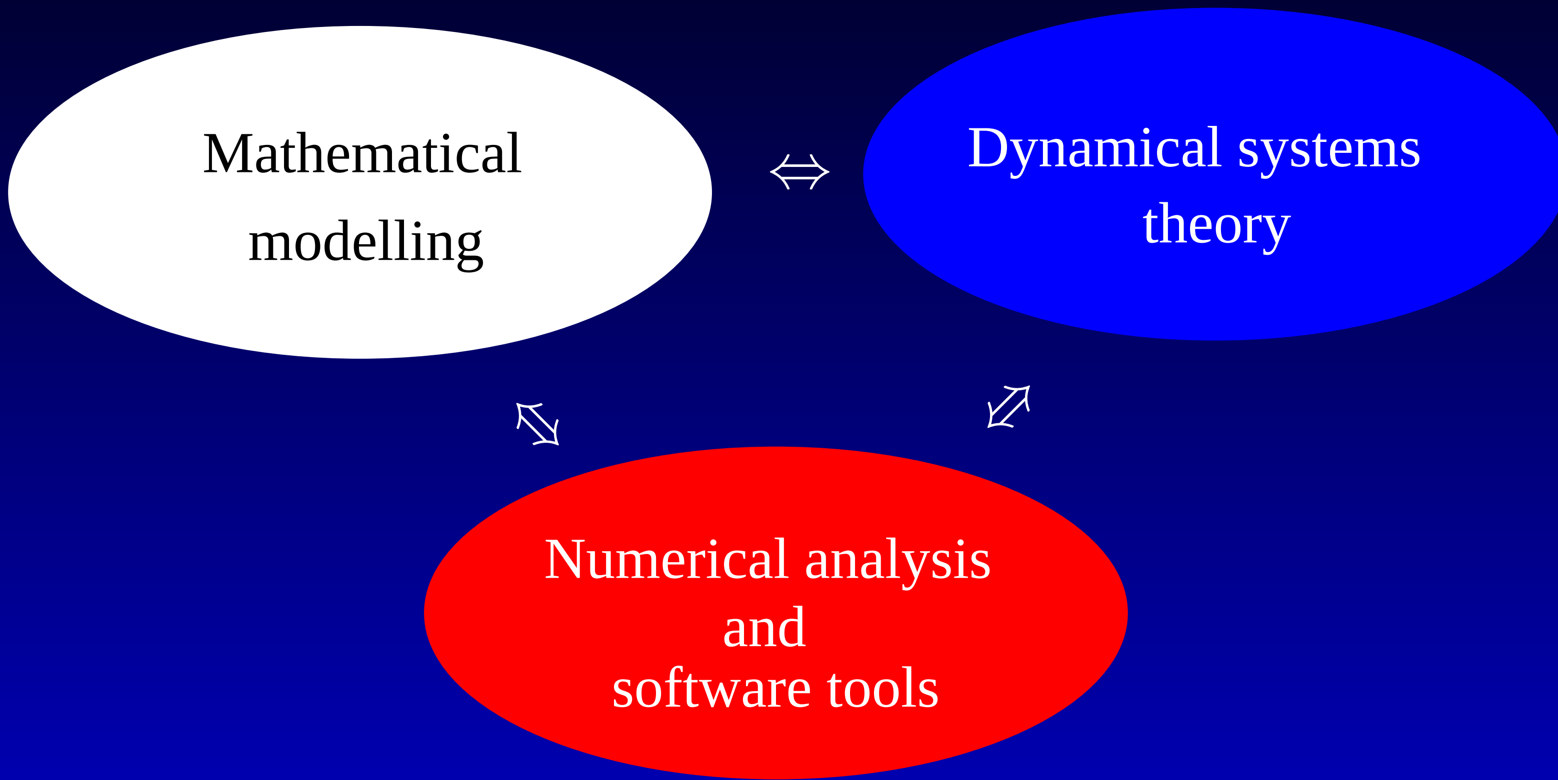
1. Mathematical analysis of deterministic systems



Example: Hodgkin-Huxley [1952] axon equations

$$\begin{cases} C\dot{V} &= I - g_{Na}m^3h(V - V_{Na}) - g_K n^4(V - V_K) - g_L(V - V_L), \\ \dot{m} &= \phi((1-m)\alpha_m - m\beta_m), \\ \dot{h} &= \phi((1-h)\alpha_h - h\beta_h), \\ \dot{n} &= \phi((1-n)\alpha_n - n\beta_n), \end{cases}$$

where

$$\phi = 3^{(T-6.3)/10},\ \psi_{\alpha_m} = (25 - V)/10,\ \psi_{\alpha_n} = (10 - V)/10,$$

$$\alpha_m = \frac{\psi_{\alpha_m}}{\exp(\psi_{\alpha_m}) - 1},\ \alpha_h = 0.07\exp(-V/20),\ \alpha_n = 0.1\frac{\psi_{\alpha_n}}{\exp(\psi_{\alpha_n}) - 1},$$

$$\beta_m = 4\exp(-V/18),\ \beta_h = \frac{1}{1 + \exp((30 - V)/10)},\ \beta_n = 0.125\exp(-V/80).$$

Other models: Connor et al. [1977], Moris-Lecar [1981]; Traub-Miles [1991]

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Numerical analysis of dynamical systems

- Simulation at fixed parameter values
  - initial-value problems;
  - spectral analysis;
  - Lyapunov exponents.
- Bifurcation analysis of parameter-dependent systems
  - stability boundaries;
  - sensitive dependence on control parameters;
  - bifurcation diagrams.

Bifurcation analysis of smooth dynamical systems

- *Continuation of orbits*:
  - Equilibria (fixed points) and cycles
  - Orbits in invariant manifolds of equilibria and cycles

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  - Normal forms for bifurcations of equilibria
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- *Branch switching at bifurcations*

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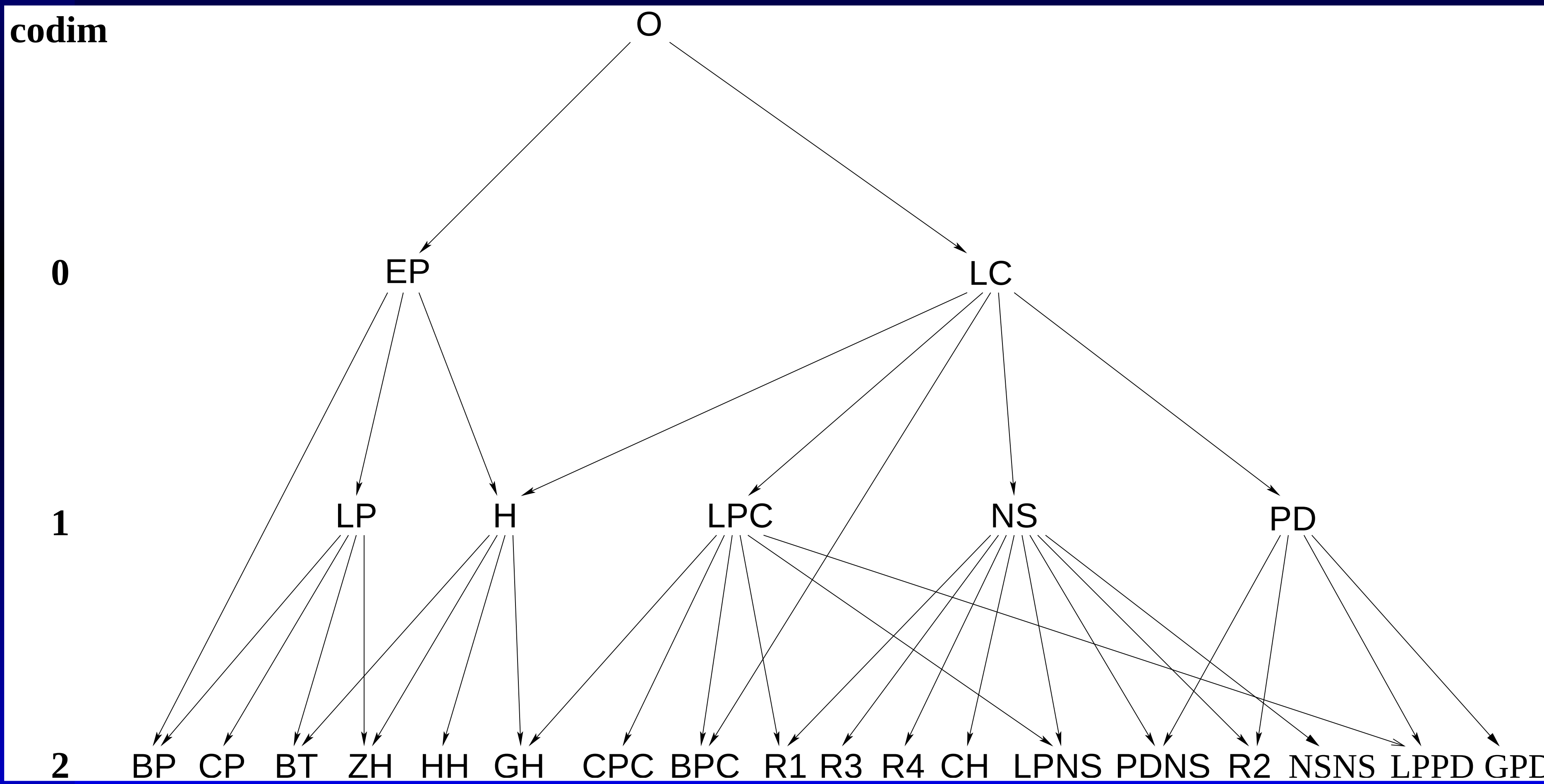
Software tools for bifurcation analysis

Standard bifurcation and continuation software:

- LOCBIF [1986-1992]
- AUTO97 (HomCont[1994-1997], SlideCont[2001-2005])
- CONTENT [1993-1998]
- MatCont [2000-]

Strategy of local bifurcation analysis of ODEs

$$\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m$$



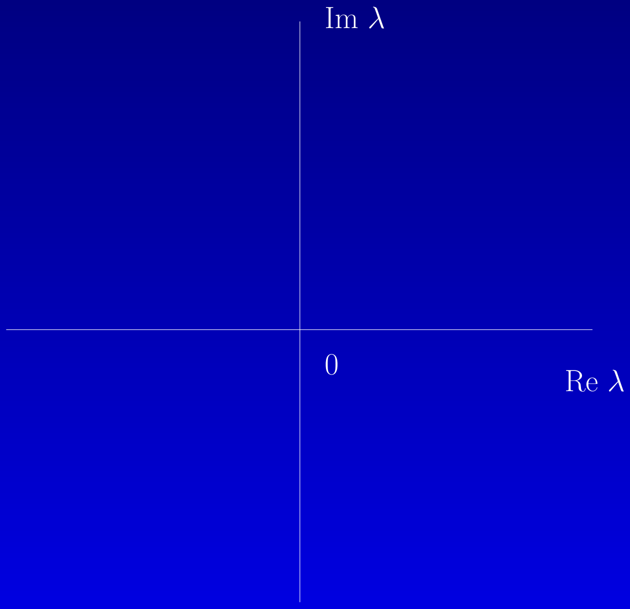
2. Equilibria of ODEs and their bifurcations

$$\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

An equilibrium x_0 satisfies

$$f(x_0, \alpha_0) = 0$$

and has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \sigma(f_x(x_0, \alpha_0))$



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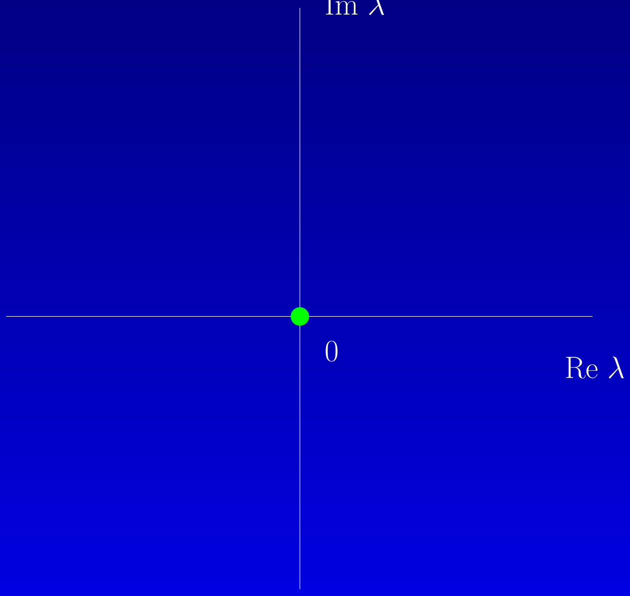

- Fold (LP): $\lambda_1 = 0$;

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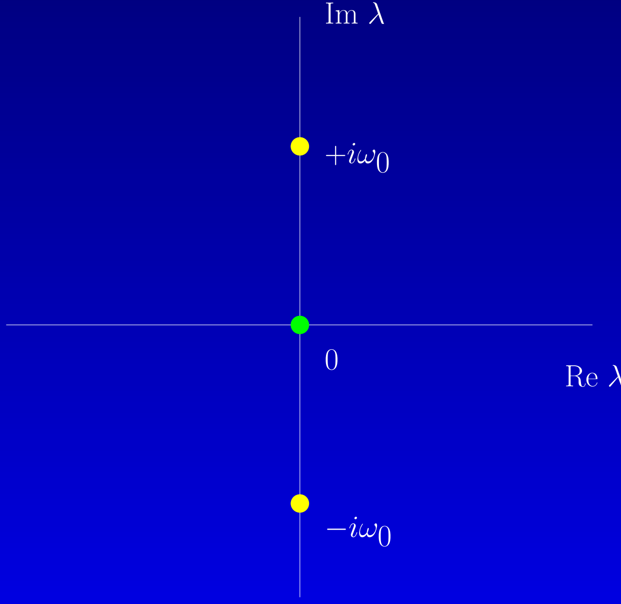
- Fold (LP): $\lambda_1 = 0$;
- Andronov-Hopf (H): $\lambda_{1,2} = \pm i\omega_0$.



Continuation of LP bifurcation in two parameters

- Defining system: $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^2$

$$\begin{cases} f(x,\alpha) & = & 0, \\ g(x,\alpha) & = & 0, \end{cases}$$

where g is computed by solving the *bordered system* [Griewank & Reddien, 1984; Govaerts, 2000]

$$\begin{pmatrix} A(x,\alpha) & w_1 \\ v_1^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $A(x,\alpha) = f_x(x,\alpha)$.

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- Vectors $v_1, w_1 \in \mathbb{R}^n$ are adapted along the LP-curve to make the linear system nonsingular.

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- Vectors $v_1, w_1 \in \mathbb{R}^n$ are adapted along the LP-curve to make the linear system nonsingular.
- (g_y, g_α) can be computed efficiently using the adjoint linear system.

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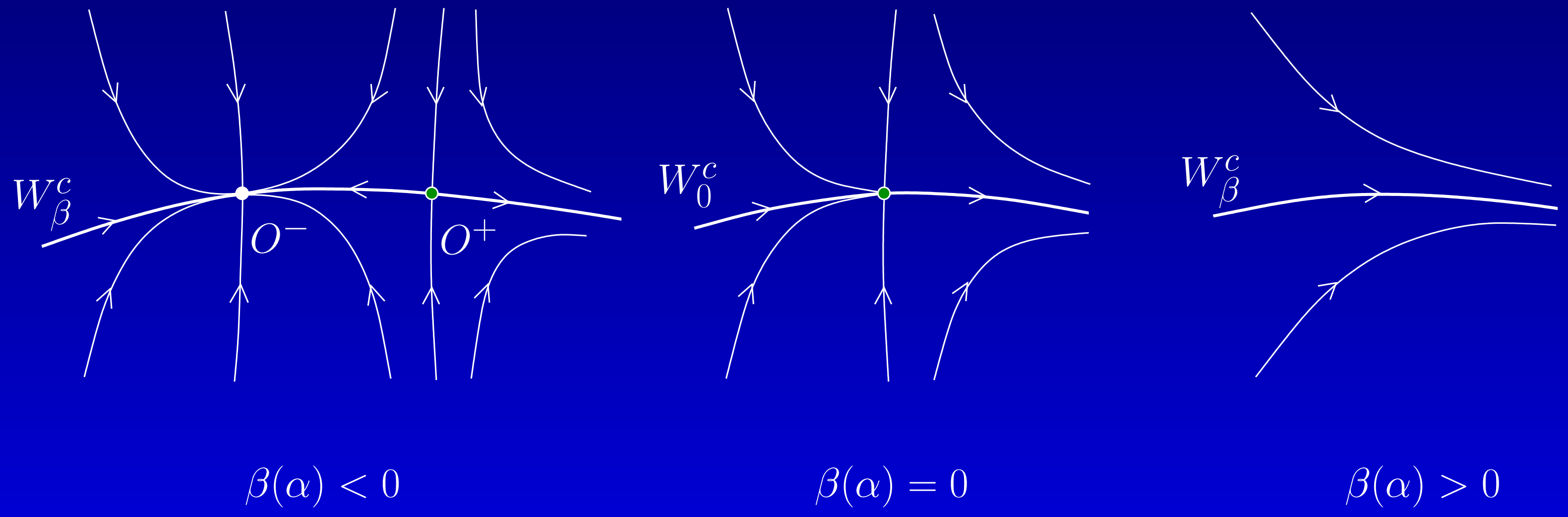
Generic fold (LP) bifurcation: $\lambda_1 = 0$

- Smooth normal form on CM:

$$\dot{\xi} = \beta + b\xi^2 + O(\xi^3), \quad b \neq 0.$$

- Topological normal form on CM:

$$\dot{\xi} = \beta + b\xi^2, \quad b \neq 0.$$



Collision and disappearance of two equilibria: $O^- + O^+ \to \emptyset$.

Critical LP-coefficient b

Write following [Coullet & Spiegel, 1983]

$$F(H) := f(x_0 + H, \alpha_0) = AH + \frac{1}{2}B(H,H) + O(\|H\|^3),$$

and locally represent the center manifold W_0^c as the graph of a function $H : \mathbb{R} \to \mathbb{R}^n$,

$$x = H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + O(\xi^3),\ \ \xi \in \mathbb{R},\ h_2 \in \mathbb{R}^n.$$

The restriction of $\dot{x} = F(x)$ to W_0^c is

$$\dot{\xi} = G(\xi) = b\xi^2 + O(\xi^3).$$

The invariance of the center manifold W_0^c implies $H_\xi \dot{\xi} = F(H(\xi))$.

$$A(\xi q + \frac{1}{2}h_2\xi^2) + \frac{1}{2}B(\xi q, \xi q) + O(|\xi|^3) = b\xi^2 q + \frac{1}{2}h_2\xi^2 + O(|\xi|^3)$$

- The ξ-terms give the identity: $Aq = 0$.
- The ξ^2-terms give the equation for h_2:

$$Ah_2 = -B(q,q) + 2bq.$$

It is singular and its *Fredholm solvability* implies

$$b = \frac{1}{2}\langle p, B(q,q)\rangle,$$

where $Aq = A^{\mathrm{T}}p = 0, \langle q, q\rangle = \langle p, q\rangle = 1.$

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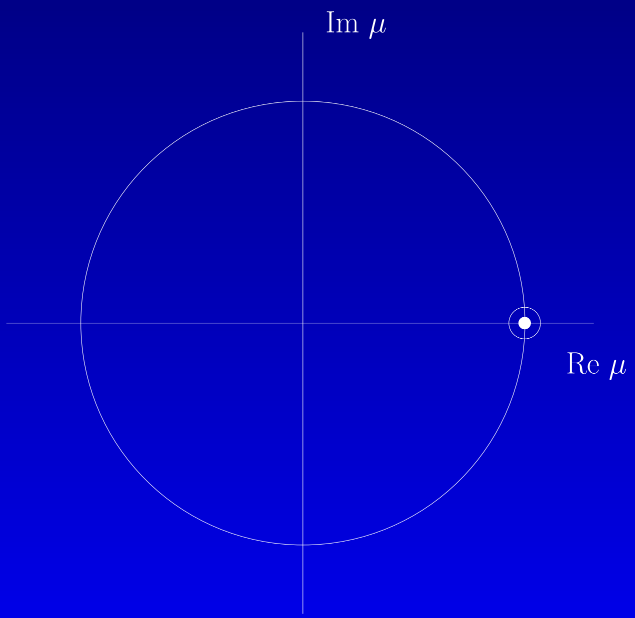
3. Limit cycles of ODEs and their local bifurcations

$$\frac{dx}{dt} = f(x, \alpha), \ \ x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

A limit cycle C_0 corresponds to a periodic solution $x_0(t + T_0) = x_0(t)$ and has Floquet multipliers $\{\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 1\} = \sigma(M(T_0))$, where

$$\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \ \ M(0) = I_n.$$



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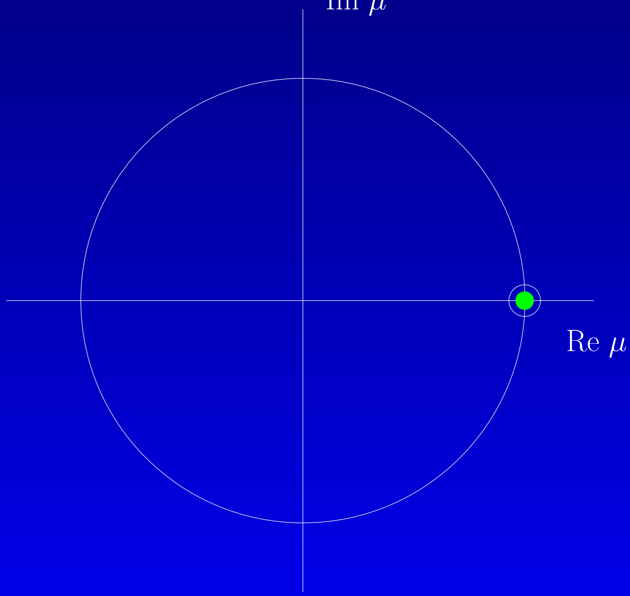
- Fold (LPC): $\mu_1 = 1$;



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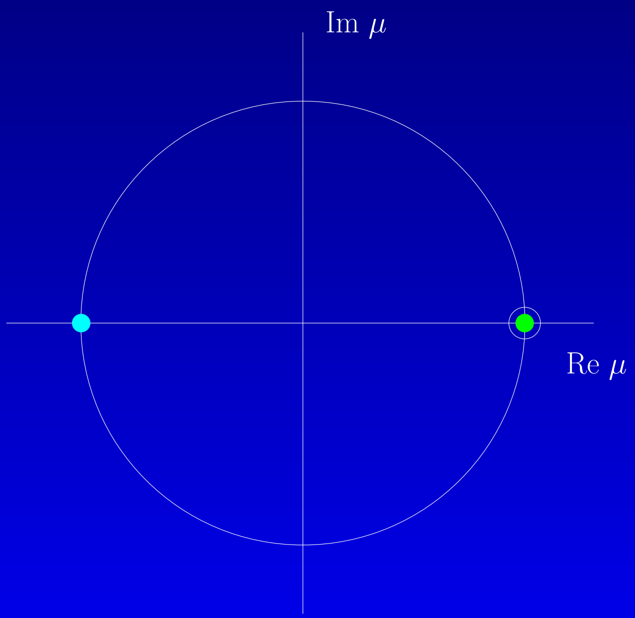
- Flip (PD): $\mu_1 = -1$;



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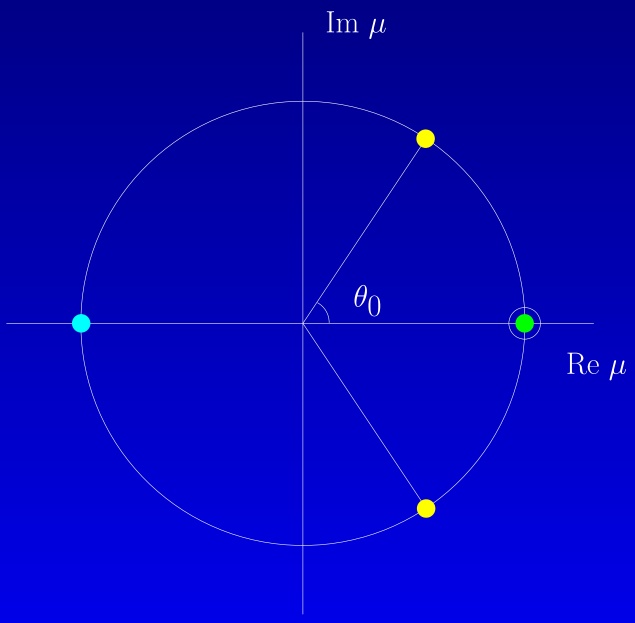
- Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}$.



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Simple bifurcation points

$$\dot\Phi(\tau) - Tf_x(u(\tau),\alpha_0)\Phi(\tau) = 0, \quad \Phi(0) = I_n,$$
$$\dot\Psi(\tau) + Tf_x^{\mathrm{T}}(u(\tau),\alpha_0)\Psi(\tau) = 0, \quad \Psi(0) = I_n.$$

- LPC:

$$(\Phi(1)-I_n)q_0 = 0, (\Phi(1)-I_n)q_1 = q_0, (\Psi(1)-I_n)p_0 = 0, (\Psi(1)-I_n)p_1$$

- PD:

$$(\Phi(1) + I_n)q_2 = 0, \ \ (\Psi(1) + I_n)p_2 = 0.$$

- NS: $\kappa = \cos\theta_0$

$$(\Phi(1) - \mathrm{e}^{i\theta_0} I_n)(q_3 + iq_4) = 0, \ \ (\Psi(1) - \mathrm{e}^{-i\theta_0} I_n)(p_3 + ip_4) = 0.$$

We have $(I_n - 2\kappa\Phi(1) + \Phi^2(1))q_{3,4} = 0.$

Continuation of bifurcations in two parameters

- PD and LPC: $(u, T, \alpha) \in C^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2$

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) &= 0, \quad \tau \in [0,1], \\ u(0) - u(1) &= 0, \\ \int_0^1 \langle \dot{\tilde{u}}(\tau), u(\tau) \rangle \, d\tau &= 0, \\ G[u, T, \alpha] &= 0. \end{cases}$$

- NS: $(u, T, \alpha, \kappa) \in C^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) &= 0, \quad \tau \in [0,1], \\ u(0) - u(1) &= 0, \\ \int_0^1 \langle \dot{\tilde{u}}(\tau), u(\tau) \rangle \, d\tau &= 0, \\ G_{11}[u, T, \alpha, \kappa] &= 0, \\ G_{22}[u, T, \alpha, \kappa] &= 0. \end{cases}$$

PD-continuation

- There exist $v_{01}, w_{01} \in C^0([0,1], \mathbb{R}^n)$, and $w_{02} \in \mathbb{R}^n$, such that $N_1 : C^1([0,1], \mathbb{R}^n) \times \mathbb{R} \to C^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$,

$$N_1 = \begin{bmatrix} D - Tf_x(u,\alpha) & w_{01} \\ \delta_0 - \delta_1 & w_{02} \\ \mathrm{Int}_{v_{01}} & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple PD bifurcation point.

- Define G by solving $\quad N_1 \begin{pmatrix} v \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

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- The BVP for (v, G) can be written in the "classical form"

$$\left\{ \begin{array}{rcl} \dot{v}(\tau) - T f_x(u(\tau), \alpha) v(\tau) + G w_{01}(\tau) & = & 0, \ \tau \in [0,1], \\ v(0) + v(1) + G w_{02} & = & 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau) \rangle d\tau - 1 & = & 0. \end{array} \right.$$

- If $G = 0$ then $\Phi(1)$ has eigenvalue $\mu_1 = -1$.
- One can take

$$w_{02} = 0$$

and

$$w_{01}(\tau) = \Psi(\tau) p_2, \ \ v_{01}(\tau) = \Phi(\tau) q_2.$$

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LPC-continuation

- There exist $v_{01}, w_{01} \in C^0([0,1],\mathbb{R}^n)$, $w_{02} \in \mathbb{R}^n$, and $v_{02}, w_{03} \in \mathbb{R}$ such that $N_2 : C^1([0,1],\mathbb{R}^n) \times \mathbb{R}^2 \to C^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

$$N_2 = \begin{bmatrix} D - Tf_x(u,\alpha) & -f(u,\alpha) & w_{01} \\ \delta_0 - \delta_1 & 0 & w_{02} \\ \mathrm{Int}_{f(u,\alpha)} & 0 & w_{03} \\ \mathrm{Int}_{v_{01}} & v_{02} & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple LPC bifurcation point.

- Define G by solving $N_2 \begin{pmatrix} v \\ S \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

NS-continuation

- There exist $v_{01}, v_{02}, w_{11}, w_{12} \in C^0([0,2], \mathbb{R}^n)$, and $w_{21}, w_{22} \in \mathbb{R}^n$, such that $N_3 : C^1([0,2], \mathbb{R}^n) \times \mathbb{R}^2 \to C^0([0,2], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

$$N_3 = \begin{bmatrix} D - Tf_x(u, \alpha) & w_{11} & w_{12} \\ \delta_0 - 2\kappa\delta_1 + \delta_2 & w_{21} & w_{22} \\ \mathrm{Int}_{v_{01}} & 0 & 0 \\ \mathrm{Int}_{v_{02}} & 0 & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple NS bifurcation point.

- Define G_{jk} by solving $\quad N_3 \begin{pmatrix} r & s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$

Remarks on continuation of bifurcations

- After discretization via orthogonal collocation, all linear BVPs for G's have sparsity structure that is identical to that of the linearization of the BVP for limit cycles.
- For each defining system holds: *Simplicity of the bifurcation + Transversality* $\Rightarrow$ *Regularity of the defining BVP*.
- Jacobian matrix of each (discretized) defining BVP can be efficiently computed using adjoint linear BVP.
- Border adaptation using solutions of the adjoint linear BVPs.
- Actually implemented in MatCont, also with compiled C-codes for the Jacobian matrices.

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Periodic normalization on center manifolds

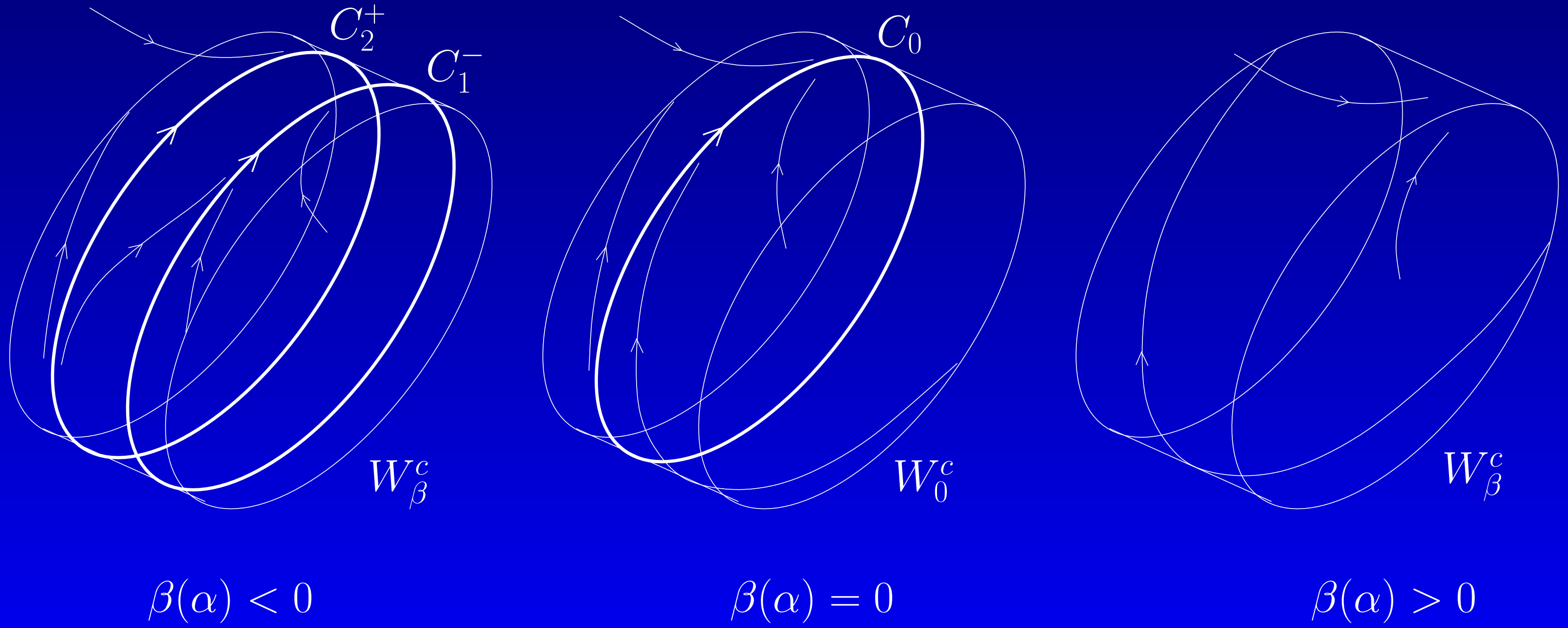
- Parameter-dependent periodic normal forms for LPC, PD, and NS [Iooss, 1988]
- Computation of critical normal form coefficients

Generic LPC-bifurcation

T_0-periodic normal form on W^c_α:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\alpha) - \xi + a(\alpha)\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} = \beta(\alpha) + b(\alpha)\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$.

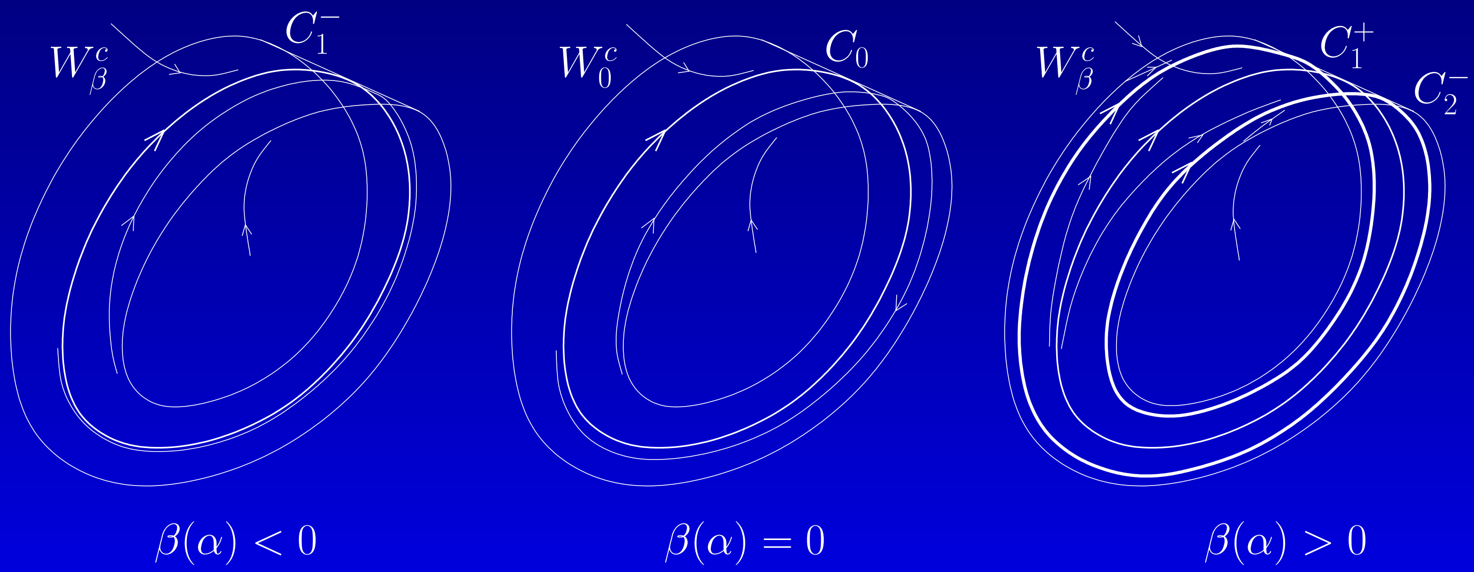


Generic PD-bifurcation

$2T_0$-periodic normal form on W^c_α:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\alpha) + a(\alpha)\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} = \beta(\alpha)\xi + c(\alpha)\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$.



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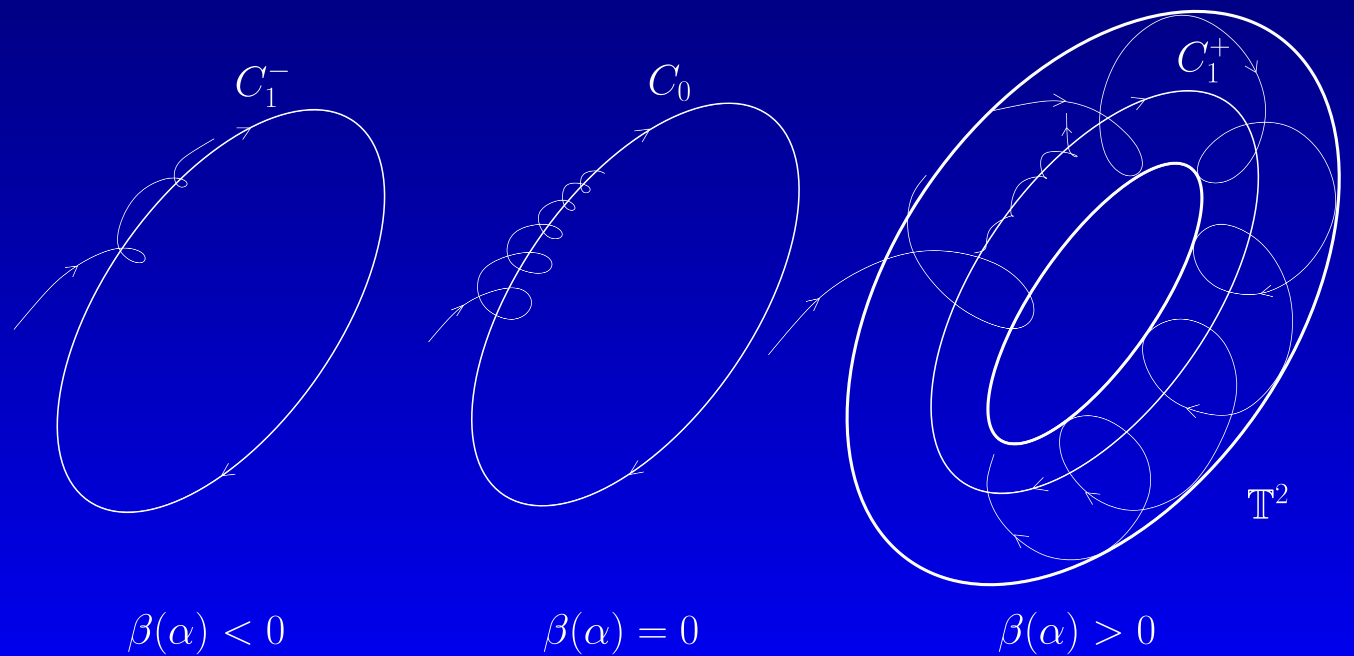
Generic NS-bifurcation

T_0-periodic normal form on W^c_α:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\alpha) + a(\alpha)|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \dfrac{d\xi}{dt} = \left(\beta(\alpha) + \dfrac{i\theta(\alpha)}{T(\alpha)}\right)\xi + d(\alpha)\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}, d \in \mathbb{C}$.



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Critical normal form coefficients

At a codimension-one point write

$$f(x_0(t)+v,\alpha_0) = f(x_0(t),\alpha_0)+A(t)v+\frac{1}{2}B(t;v,v)+\frac{1}{6}C(t;v,v,v)+O(\|v\|^4),$$

where $A(t) = f_x(x_0(t),\alpha_0)$ and the components of the multilinear functions B and C are given by

$$B_i(t;u,v) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 f_i(x,\alpha_0)}{\partial x_j \partial x_k}\right|_{x=x_0(t)} u_j v_k$$

and

$$C_i(t;u,v,w) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 f_i(x,\alpha_0)}{\partial x_j \partial x_k \partial x_l}\right|_{x=x_0(t)} u_j v_k w_l,$$

for $i = 1,2,\ldots,n$. These are T_0-periodic in t.

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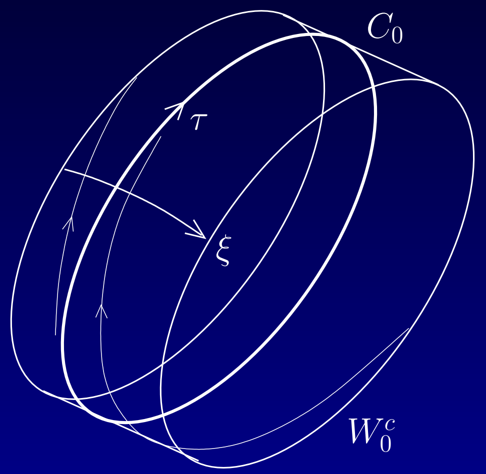
Fold (LPC): $\mu_1 = 1$

- Critical center manifold $W_0^c : \tau \in [0, T_0],\ \xi \in \mathbb{R}$



$$x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),$$

where $H(T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau) \xi^2 + O(\xi^3)$$

- Critical periodic normal form on W_0^c:

$$\begin{cases} \frac{d\tau}{dt} & = & 1 - \xi + a\xi^2 + \mathcal{O}(\xi^3), \\ \frac{d\xi}{dt} & = & b\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, while the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

LPC: Eigenfunctions

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) - f(x_0(\tau), \alpha_0) &= 0, \ \tau \in [0, T_0], \\ v(0) - v(T_0) &= 0, \\ \int_0^{T_0} \langle v(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau &= 0, \end{cases}$$

implying

$$\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle \, d\tau = 0,$$

where φ^* satisfies

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) &= 0, \ \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) &= 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), v(\tau) \rangle d\tau - 1 &= 0. \end{cases}$$

LPC: Computation of b

- Substitute into

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial x}{\partial \tau}\frac{d\tau}{dt}$$

- Collect

$$\begin{aligned} \xi^0 &: \quad \dot{x}_0 = f(x_0, \alpha_0), \\ \xi^1 &: \quad \dot{v} - A(\tau)v = \dot{x}_0, \\ \xi^2 &: \quad \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2af(x_0, \alpha_0) + 2\dot{v} - 2bv. \end{aligned}$$

- *Fredholm solvability condition*

$$b = \frac{1}{2}\int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau)\rangle \, d\tau.$$

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Remarks on numerical periodic normalization

- Only the derivatives of $f(x, \alpha_0)$ are used, not those of the Poincaré map $\mathcal{P}(y, \alpha_0)$.
- Detection of codim 2 points is easy.
- After discretization via orthogonal collocation, all linear BVPs involved have the standard sparsity structure.
- One can re-use solutions to linear BVPs appearing in the continuation to compute the normal form coefficients.
- Actually implemented in MatCont for LPC, PD, and NS.

Example: Oscillations in peroxidase-oxidase reaction

- $2YH_2 + O_2 + 2H^+ \rightarrow 2YH^+ + 2H_2O$

$$
\begin{aligned}
A + B + X &\xrightarrow{k_1} 2X, & Y &\xrightarrow{k_5} Q, \\
2X &\xrightarrow{k_2} 2Y, & X_0 &\xrightarrow{k_6} X, \\
A + B + Y &\xrightarrow{k_3} 2X, & A_0 &\overset{k_7}{\leftrightarrow} A, \\
X &\xrightarrow{k_4} P, & B_0 &\xrightarrow{k_8} B
\end{aligned}
$$

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Example: Oscillations in peroxidase-oxidase reaction

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X &\xrightarrow{k_4} P, & B_0 &\xrightarrow{k_8} B
\end{aligned}
$$

- Steinmetz & Larter (1991):

$$
\begin{cases}
\dot{A} = -k_1ABX - k_3ABY + k_7 - k_{-7}A,\\
\dot{B} = -k_1ABX - k_3ABY + k_8,\\
\dot{X} = k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6,\\
\dot{Y} = -k_3ABY + 2k_2X^2 - k_5Y.
\end{cases}
$$

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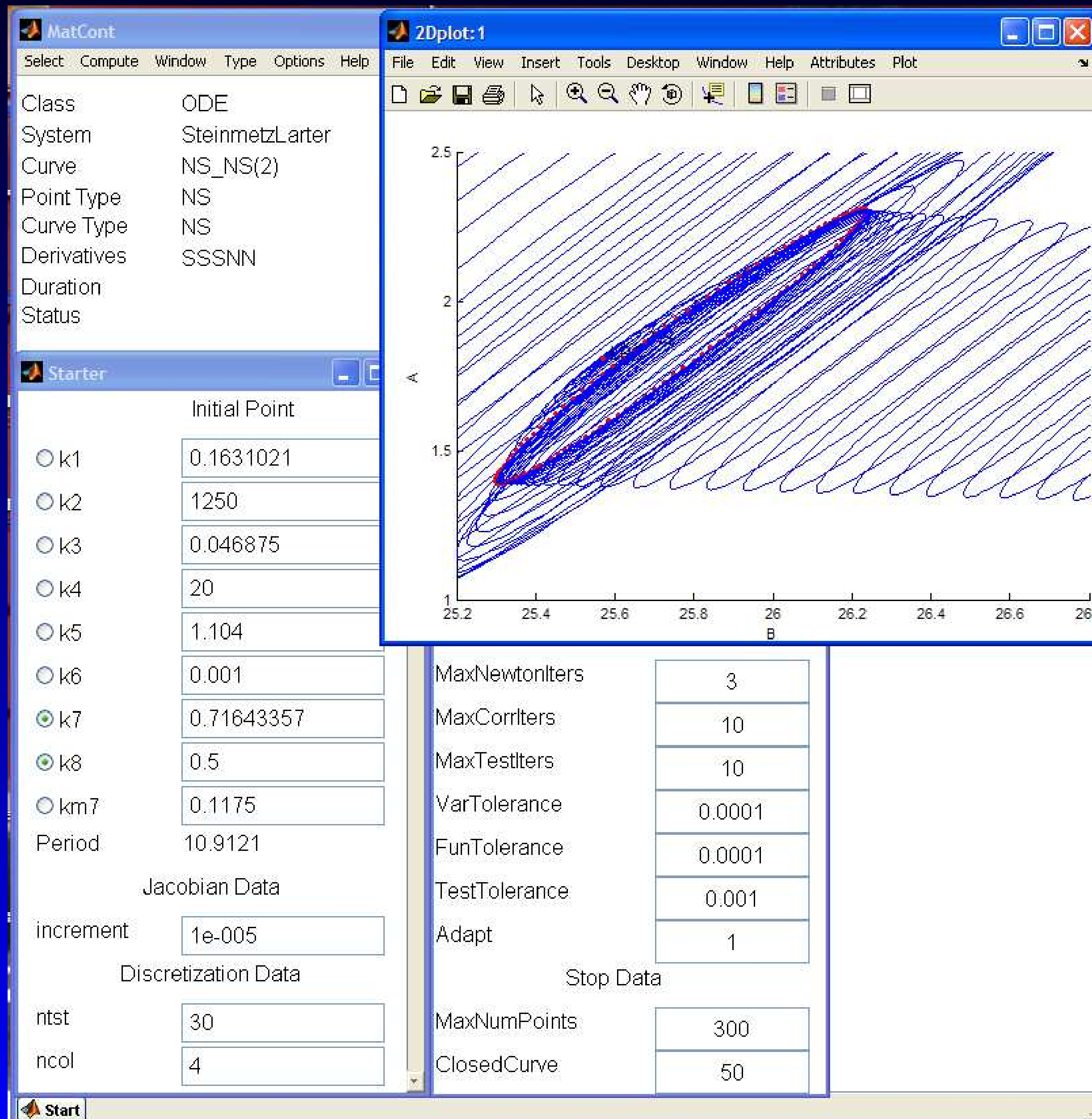
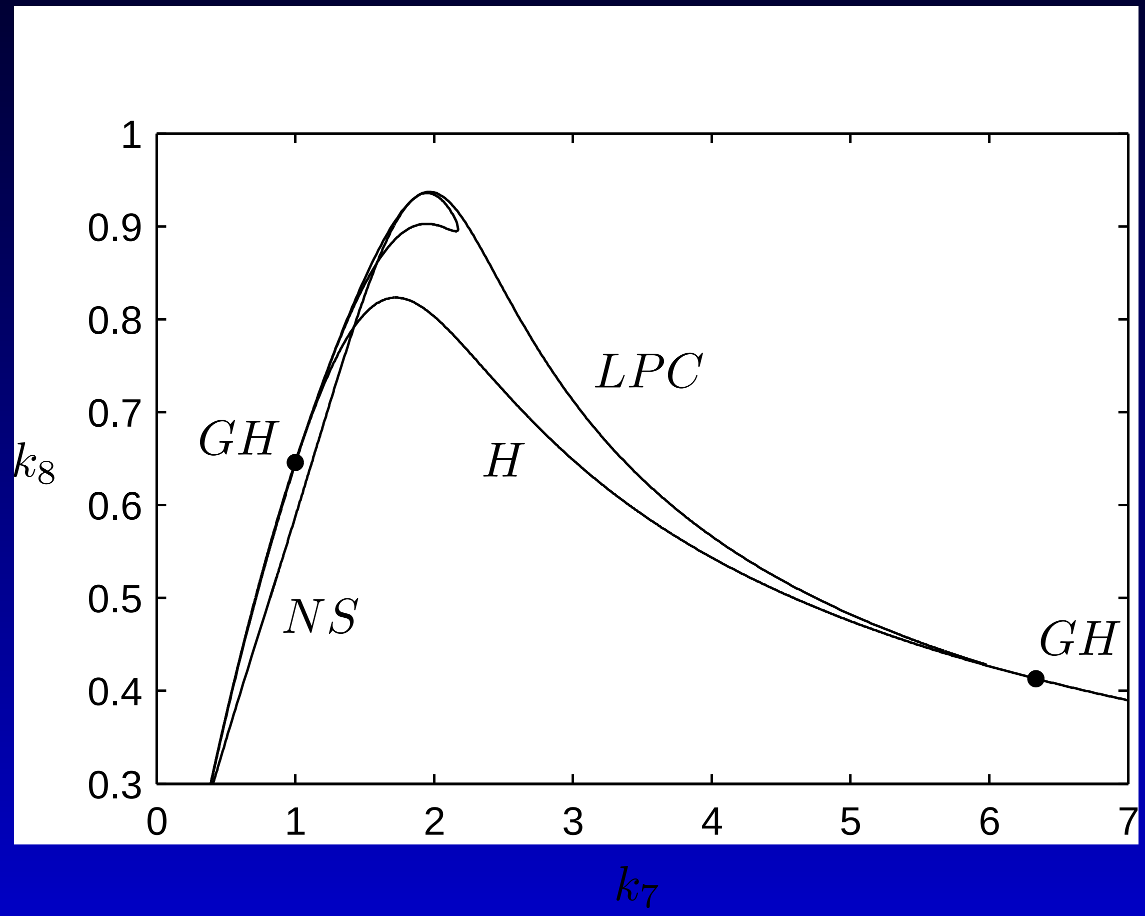
MatCont


MatCont
Select Compute Window Type Options Help
Class ODE
System SteinmetzLarter
Curve NS_NS(2)
Point Type NS
Curve Type NS
Derivatives SSSNN
Duration
Status
Starter
Initial Point
k1 0.1631021
k2 1250
k3 0.046875
k4 20
k5 1.104
k6 0.001
k7 0.71643357
k8 0.5
km7 0.1175
Period 10.9121
Jacobian Data
increment 1e-005
Discretization Data
ntst 30
ncol 4
2Dplot:1
File Edit View Insert Tools Desktop Window Help Attributes Plot
2.5
2
1.5
1
25.2
25.4
25.6
25.8
26
26.2
26.4
26.6
B
MaxNewtonIters 3
MaxCorrIters 10
MaxTestIters 10
VarTolerance 0.0001
FunTolerance 0.0001
TestTolerance 0.001
Adapt 1
Stop Data
MaxNumPoints 300
ClosedCurve 50
Start

Bifurcation curves

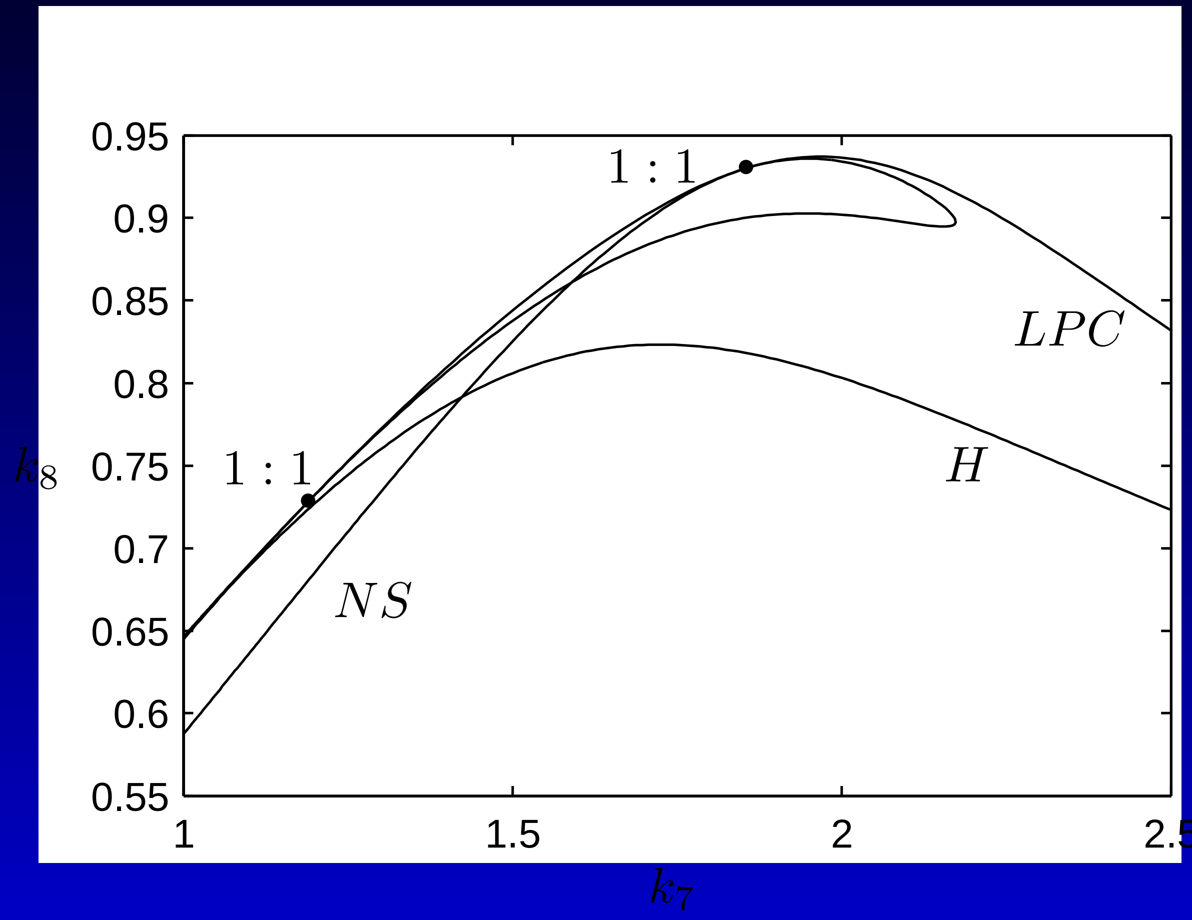


ACM Trans. Math. Software **24** (1998), 418-436

SIAM J. Numer. Anal. **38** (2000), 329-346

SIAM J. Sci. Comp. **27** (2005), 231-252

Bifircation curves (zoom)



SIAM J. Numer. Anal. **41** (2003), 401-435

SIAM J. Numer. Anal. **43** (2005), 1407-1435

Physica D **237** (2008), 3061-3068

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4. Bifurcations of homoclinic orbits

- Consider a family of smooth ODEs

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^n, \;\; \alpha \in \mathbb{R}^m,$$

  having a hyperbolic equilibrium x_0 with eigenvalues

$$\Re(\mu_{n_S}) \leq \ldots \leq \Re(\mu_1) < 0 < \Re(\lambda_1) \leq \ldots \leq \Re(\lambda_{n_U})$$

  of $A(x_0,\alpha) = f_x(x_0,\alpha)$.

- Homoclinic problem:

$$\begin{cases} f(x_0,\alpha) &= 0, \\ \dot{x}(t) - f(x(t),\alpha) &= 0, \\ \lim_{t\to\pm\infty} x(t) - x_0 &= 0, \; t \in \mathbb{R}, \\ \int_{-\infty}^{\infty} \dot{\tilde{x}}(t)^{\mathrm{T}}(x(t) - \tilde{x}(t))dt &= 0, \end{cases}$$

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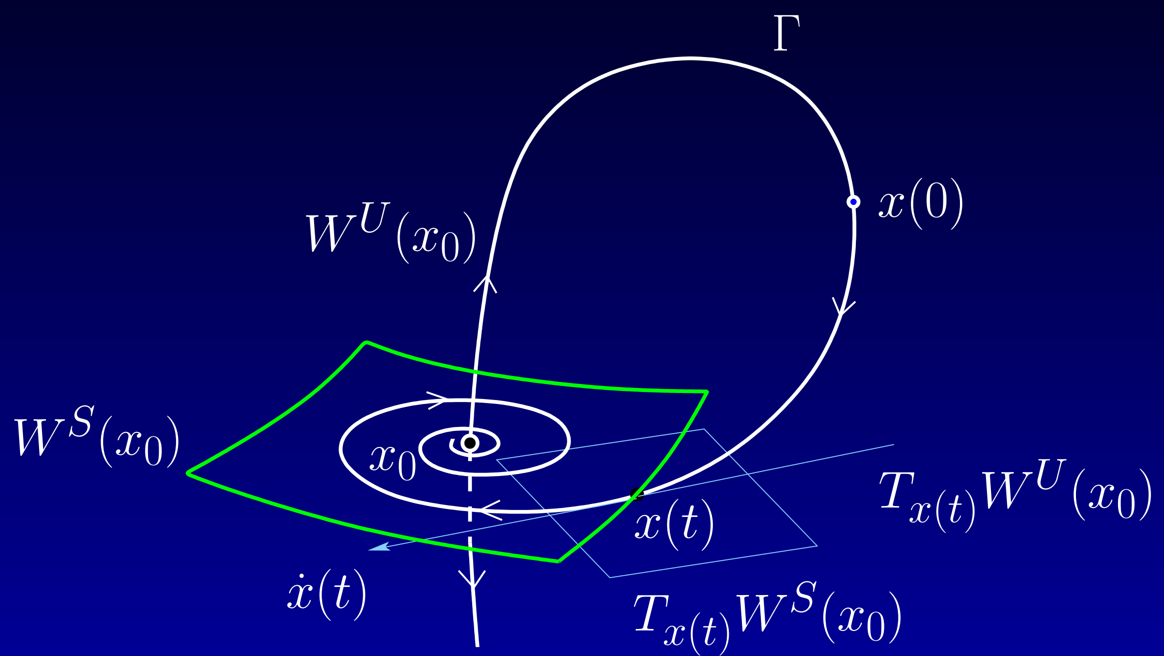
Homoclinic orbits



Homoclinic orbits to hyperbolic equilibria have codim 1.

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Defining BVP [Beyn, 1993; Doedel & Friedman 1994]

- Truncate with *projection boundary conditions*:

$$
\begin{cases}
f(x_0,\alpha) & = & 0, \\
\dot{x}(t) - f(x(t),\alpha) & = & 0, \ t \in [-T,T] \\
\langle x(-T) - x_0, q_{0,n_U+i} \rangle & = & 0, \ i = 1,2,\ldots,n_S \\
\langle x(+T) - x_0, q_{1,n_S+i} \rangle & = & 0, \ i = 1,2,\ldots,n_U \\
\int_{-T}^{T} \dot{\tilde{x}}(t)^{\mathrm{T}}(x(t) - \tilde{x}(t))dt & = & 0,
\end{cases}
$$

where the columns of $Q^{U^\perp} = [q_{0,n_U+1},\ldots,q_{0,n_U+n_S}]$ and $Q^{U^\perp} = [q_{1,n_S+1},\ldots,q_{1,n_S+n_U}]$ span the orthogonal complements to $T_{x_0}W^U(x_0)$ and $T_{x_0}W^S(x_0)$, resp.

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Smooth Schur Block Factorization

$$A(s) = Q(s) \begin{bmatrix} R_{11}(s) & R_{12}(s) \\ 0 & R_{22}(s) \end{bmatrix} Q^{\mathrm{T}}(s) \in \mathbb{R}^{n \times n},$$

where $Q(s) = [Q_1(s) \;\; Q_2(s)]$ such that

- $Q(s)$ is orthogonal, i.e. $Q^{\mathrm{T}}(s)Q(s) = I_n$;
- the columns of $Q_1(s) \in \mathbb{R}^{n \times m}$ span an eigenspace $\mathcal{E}(s)$ of $A(s)$;
- the columns of $Q_2(s) \in \mathbb{R}^{n \times (n-m)}$ span $\mathcal{E}^{\perp}(s)$;
- eigenvalues of R_{11} are the eigenvalues of $A(s)$ corresponding to $\mathcal{E}(s)$;
- $Q(s)$ and $R_{ij}(s)$ have the same smoothness as $A(s)$.

Then holds the *invariant subspace relation*:

$$Q_2^{\mathrm{T}}(s)A(s)Q_1(s) = 0.$$

CIS-algorithm [Dieci & Friedman, 2001]

- Define
$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} = Q^{\mathrm{T}}(0)A(s)Q(0)$$
for small $|s|$, where $T_{11}(s) \in \mathbb{R}^{m\times m}$.
- Compute $Y \in \mathbb{R}^{(n-m)\times m}$ from the *Riccati matrix equation*
$$YT_{11}(s) - T_{22}(s)Y + YT_{12}(s)Y = T_{21}(s).$$
- Then $Q(s) = Q(0)U(s)$ where $U(s) = [U_1(s)\;\; U_2(s)]$ with

$$U_1(s) = \begin{pmatrix} I_m \\ Y \end{pmatrix} (I_{n-m}+Y^{\mathrm{T}}Y)^{-\frac{1}{2}},\; U_2(s) = \begin{pmatrix} -Y^{\mathrm{T}} \\ I_{n-m} \end{pmatrix} (I_{n-m}+YY^{\mathrm{T}})^{-\frac{1}{2}}$$

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- The columns of

$$Q_1(s) = Q(0)U_1(s) \quad \text{and} \quad Q_2(s) = Q(0)U_2(s)$$

form *orthogonal* bases in $\mathcal{E}(s)$ and $\mathcal{E}^\perp(s)$.

- The columns of

$$Q(0)\begin{bmatrix} I_m \\ Y(s) \end{bmatrix} \quad \text{and} \quad Q(0)\begin{bmatrix} -Y(s)^{\mathrm{T}} \\ I_{n-m} \end{bmatrix}$$

form bases in $\mathcal{E}(s)$ and $\mathcal{E}^\perp(s)$, which are in general *non-orthogonal*.

Continuation of homoclinic orbits in MatCont

$$
\left\{
\begin{array}{rcl}
\dot{x}(t) - 2Tf(x(t),\alpha) & = & 0, \\
f(x_0,\alpha) & = & 0, \\
\int_0^1 \dot{\widetilde{x}}(t)^{\mathrm{T}}(x(t)-\widetilde{x}(t))dt & = & 0, \\
\langle x(0)-x_0, q_{0,n_U+i}\rangle & = & 0,\ i=1,2,\ldots,n_S \\
\langle x(1)-x_0, q_{1,n_S+i}\rangle & = & 0,\ i=1,2,\ldots,n_U \\
T_{22U}Y_U - Y_U T_{11U} + T_{21U} - Y_U T_{12U} Y_U & = & 0, \\
T_{22S}Y_S - Y_S T_{11S} + T_{21S} - Y_S T_{12S} Y_S & = & 0, \\
\|x(0)-x_0\| - \epsilon_0 & = & 0, \\
\|x(1)-x_0\| - \epsilon_1 & = & 0,
\end{array}
\right.
$$

$$
\begin{array}{rcl}
[q_{0,n_U+1}\ q_{0,n_U+2}\ \cdots\ q_{0,n_U+n_S}] & = & Q_U(0)\begin{bmatrix} -Y_U^{\mathrm{T}} \\ I_{n_S} \end{bmatrix} \\
[q_{1,n_S+1}\ q_{1,n_S+2}\ \cdots\ q_{1,n_S+n_U}] & = & Q_S(0)\begin{bmatrix} -Y_S^{\mathrm{T}} \\ I_{n_U} \end{bmatrix}.
\end{array}
$$

Example: Complex nerve pulses

- The slow subsystem of the Hodgkin-Huxley PDEs is approximated by the FitHugh-Nagumo [1962] system

$$\begin{cases} u_t &= u_{xx} - u(u-a)(u-1) - v, \\ v_t &= bu, \end{cases}$$

where $0 < a < 1, b > 0$.

Example: Complex nerve pulses

- The slow subsystem of the Hodgkin-Huxley PDEs is approximated by the FitHugh-Nagumo [1962] system

$$\begin{cases} u_t &= u_{xx} - u(u-a)(u-1) - v, \\ v_t &= bu, \end{cases}$$

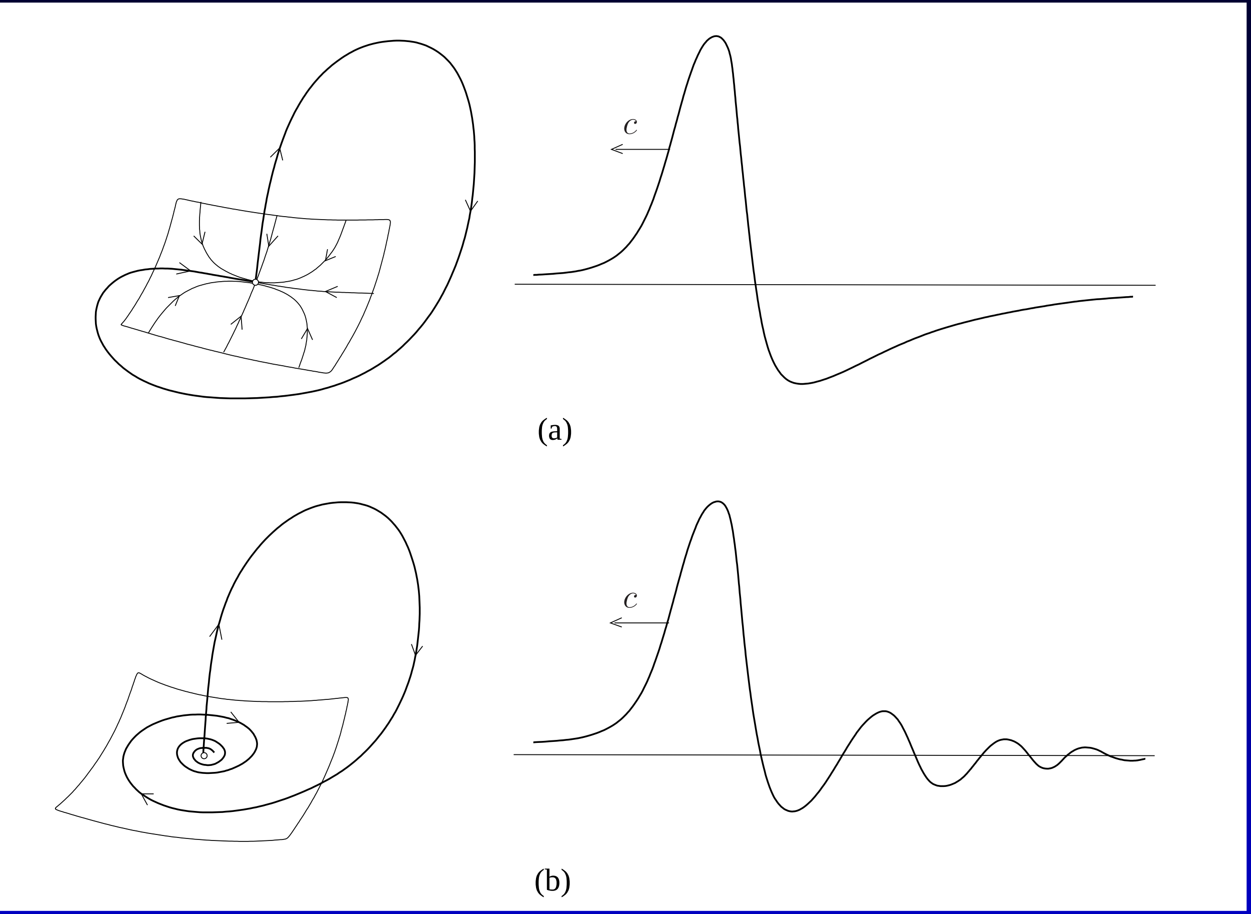
where $0 < a < 1, b > 0$.

- Traveling waves $u(x,t) = U(\xi), v(x,t) = V(\xi)$ with $\xi = x + ct$ satisfy

$$\begin{cases} \dot{U} &= W, \\ \dot{W} &= cW + U(U-a)(U-1) + V, \\ \dot{V} &= \frac{b}{c}U, \end{cases}$$

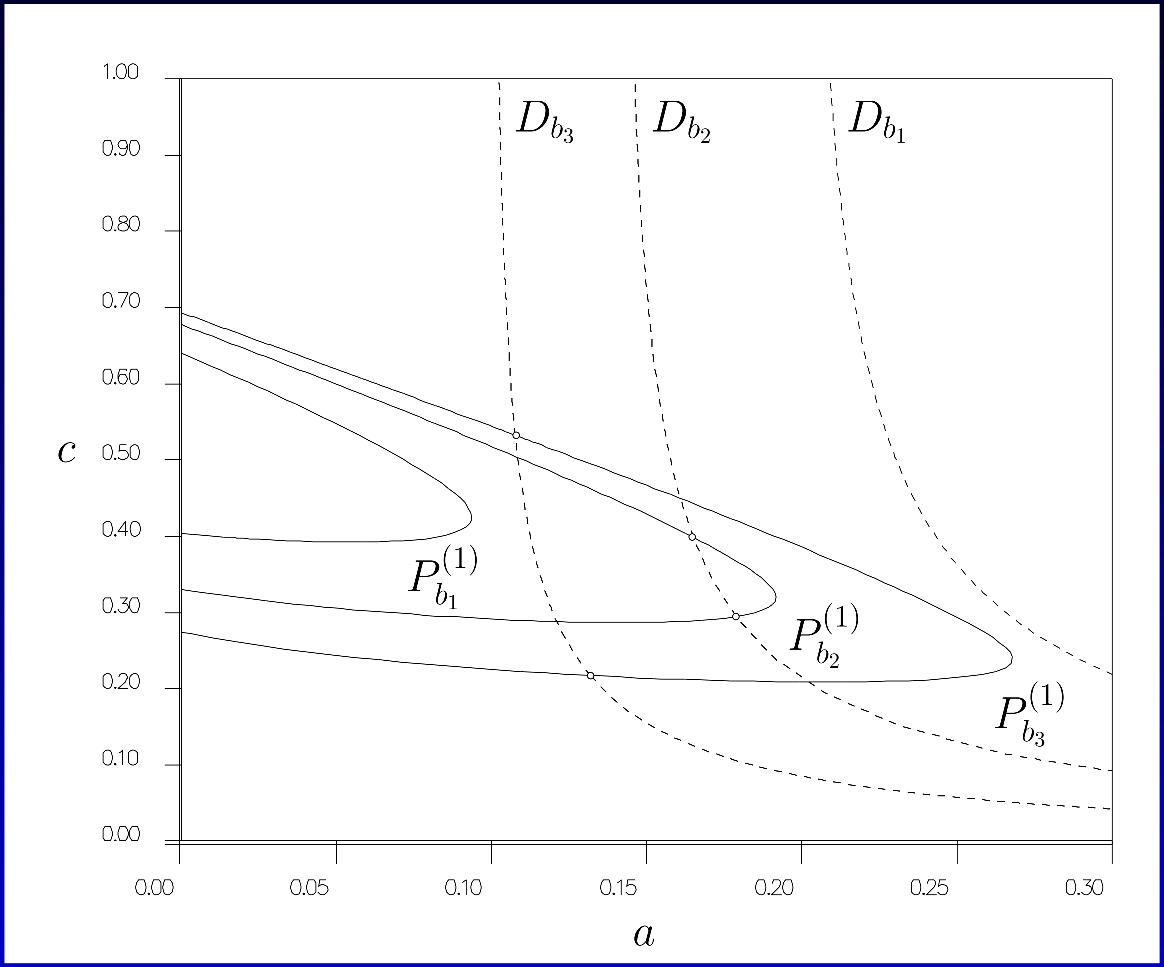
where c is the propagation speed.

Homoclinic orbits define traveling impulses



(a) (b)

Homoclinic bifurcation curves (HomCont/MatCont)



Int. J. Bifurcation & Chaos **4** (1994), 795-822

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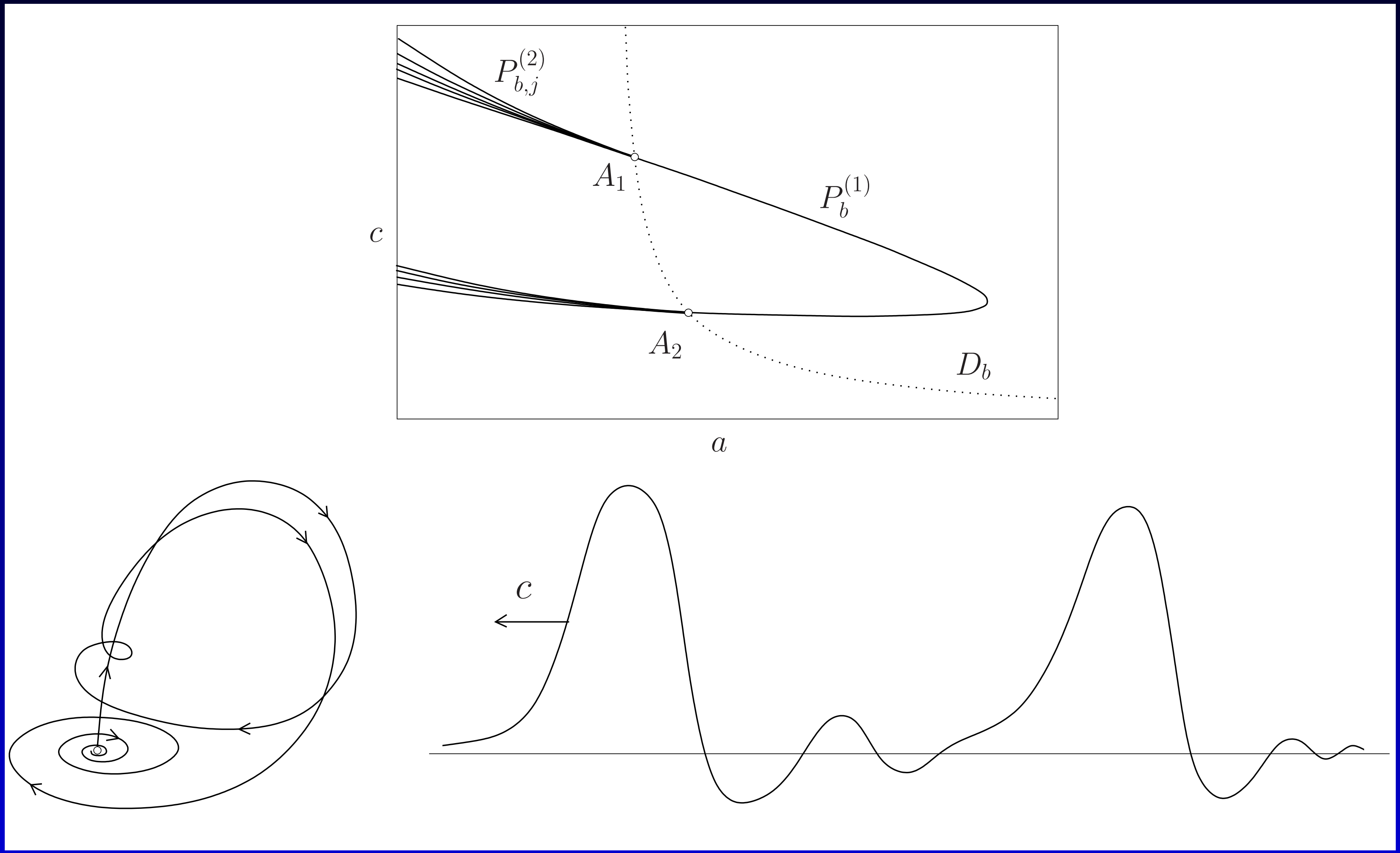
Double pulses



Selecta Math. Sovetica **13** (1994), 128-142
SIAM J. Appl. Math. **62** (2001), 462-487

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5. Open problems

- Computation of normal forms for bifurcations of equilibria in ODEs with delays.
- Bifurcation analysis of spatially-distributed systems with delays, e.g. neural fields.
- Computing codim 2 periodic normal forms for limit cycles.
- Location and continuation of homoclinic and heteroclinic orbits to limit cycles.
- Starting homoclinic codim 1 bifurcations from codim 2 points.
- ...

Example: Delay-differential equations

- Model of two interacting layers of neurons [Visser et al., 2010]:

$$\begin{cases} \dot{x}_1(t) &= -x_1(t) - aG(bx_1(t-\tau_1)) + cG(dx_2(t-\tau_2)), \\ \dot{x}_2(t) &= -x_2(t) - aG(bx_2(t-\tau_1)) + cG(dx_1(t-\tau_2)), \end{cases}$$

  where $G(x) = (\tanh(x-1) + \tanh(1))\cosh^2(1)$ and x_j is the population averaged neural activity in layer $j = 1, 2$.

- For $b = 2, d = 1.2, \tau_1 = 12.99, \tau_2 = 20.15$ there is a *double Hopf* (HH) bifurcation at

$$(abG'(0), cdG'(0)) = (0.559667, 0.688876)$$

  that gives rise to a stable quasi-periodic behaviour with two base frequences (2-*torus*).

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Example: Neural field model

The two-population neural network model:

$$\begin{cases} \dfrac{1}{\alpha_E}\dfrac{\partial u_E(x,t)}{\partial t} &= -u_E(x,t) \\ &+ \int_{-\infty}^{\infty} w_{EE}(y) f_E(u_E(x-y, t-\dfrac{|y|}{v_E}))\, dy \\ &- \int_{-\infty}^{\infty} w_{EI}(y) f_I(u_I(x-y, t-\dfrac{|y|}{v_I}))\, dy, \\ \dfrac{1}{\alpha_I}\dfrac{\partial u_I(x,t)}{\partial t} &= -u_I(x,t) \\ &+ \int_{-\infty}^{\infty} w_{IE}(y) f_E(u_E(x-y, t-\dfrac{|y|}{v_E}))\, dy \\ &- \int_{-\infty}^{\infty} w_{II}(y) f_I(u_I(x-y, t-\dfrac{|y|}{v_I}))\, dy. \end{cases}$$

Turing instabilities and pattern formation, cf. [Blomquist et al., 2005; Venkov & Coombes, 2007; Wyller et al., 2007]. Partial results but no systematic bifurcation analysis.

Cusp bifurcation of limit cycles (codim 2)

- Critical center manifold $W_0^c : \tau \in [0, T_0],\ \xi \in \mathbb{R}$

$$x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),$$

where $H(T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + O(\xi^3)$$

- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 - \xi + a_1\xi^2 + a_2\xi^3 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} = e\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a_{1,2}, e \in \mathbb{R}$, while the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

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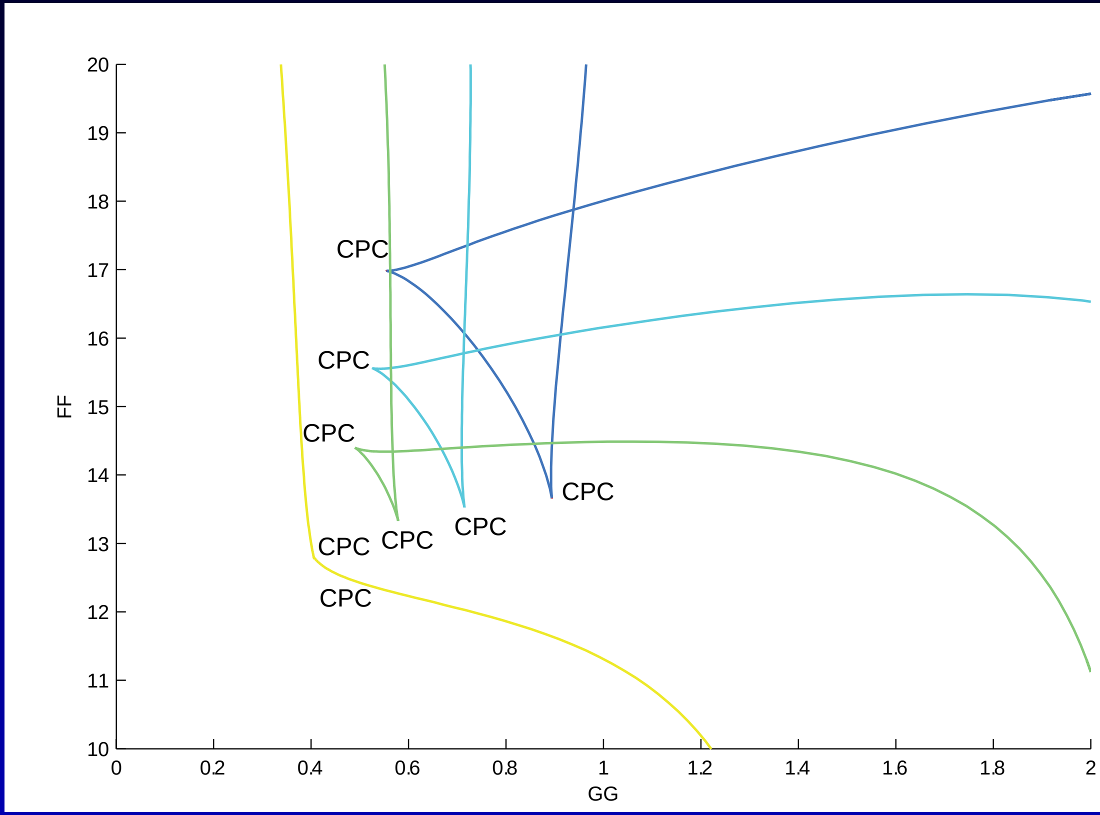
Example: Swallow-tail bifurcation in Lorenz-84

$$
\begin{cases}
\dot{x} = -y^2 - z^2 - ax + aF, \\
\dot{y} = xy - bxz - y + G, \\
\dot{z} = bxy + xz - z.
\end{cases}
$$



$$a = 0.25, \quad b \in [2.95, 4.0].$$

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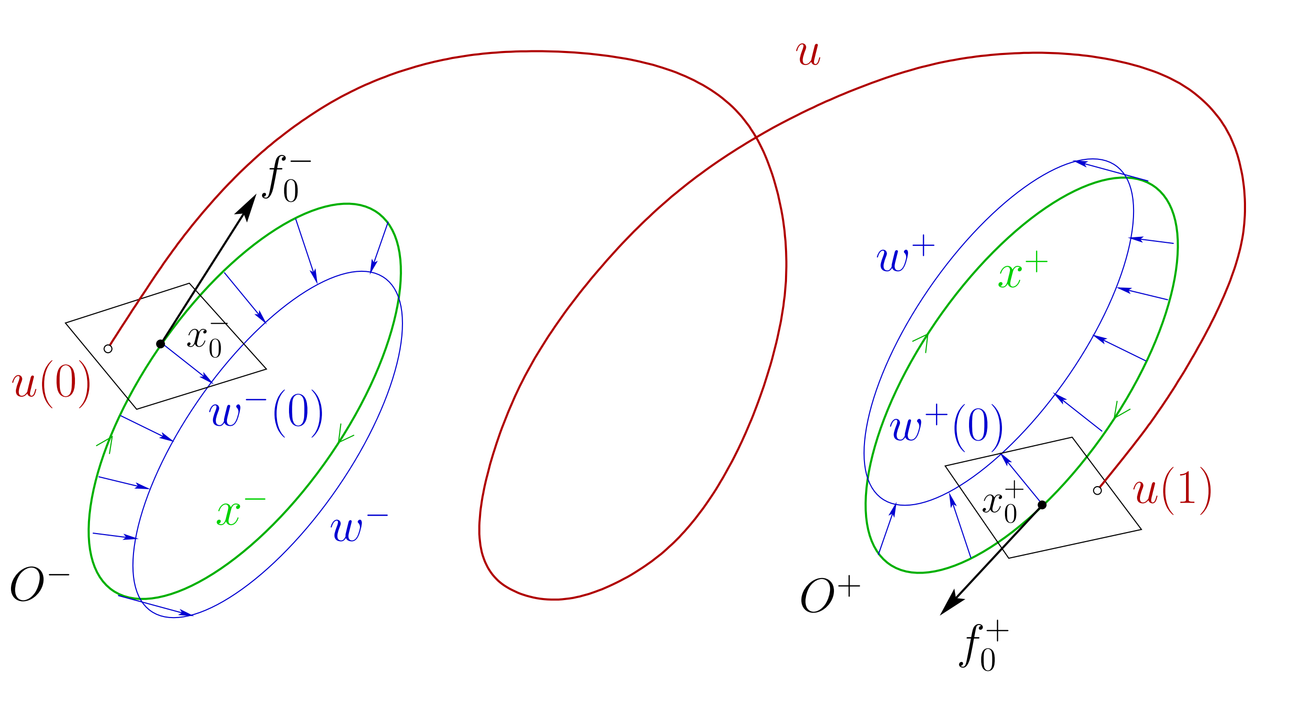
Computation of cycle-to-cycle connecting orbits in 3D



- ODEs for both cycles, their (adjoint) eigenfunctions, and the connection;
- Projection boundary conditions in orthogonal planes at base points.

Example: Bifurcations and chaos in ecology

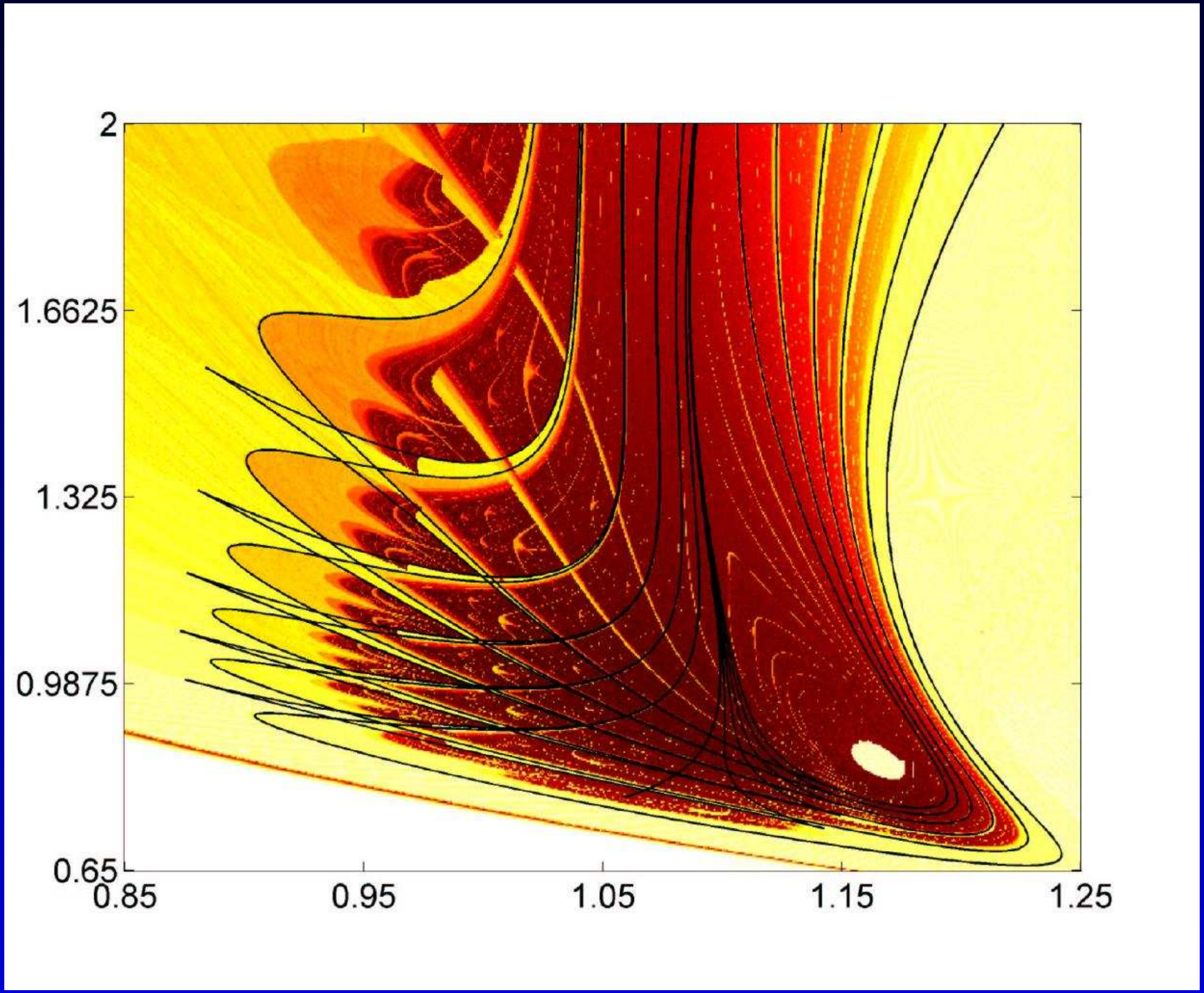
- The tri-trophic food chain model [Hogeweg & Hesper, 1978]:

$$\begin{cases} \dot{x}_1 & = & r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{a_1 x_1 x_2}{1 + b_1 x_1}, \\ \dot{x}_2 & = & e_1 \frac{a_1 x_1 x_2}{1 + b_1 x_1} - \frac{a_2 x_2 x_3}{1 + b_2 x_2} - d_1 x_2, \\ \dot{x}_3 & = & e_2 \frac{a_2 x_2 x_3}{1 + b_2 x_2} - d_2 x_3, \end{cases}$$

  where

x_1	prey biomass
x_2	predator biomass
x_3	super−predator biomass

Global bifurcation diagram



SIAM J. Appl. Math. **62** (2001), 462-487

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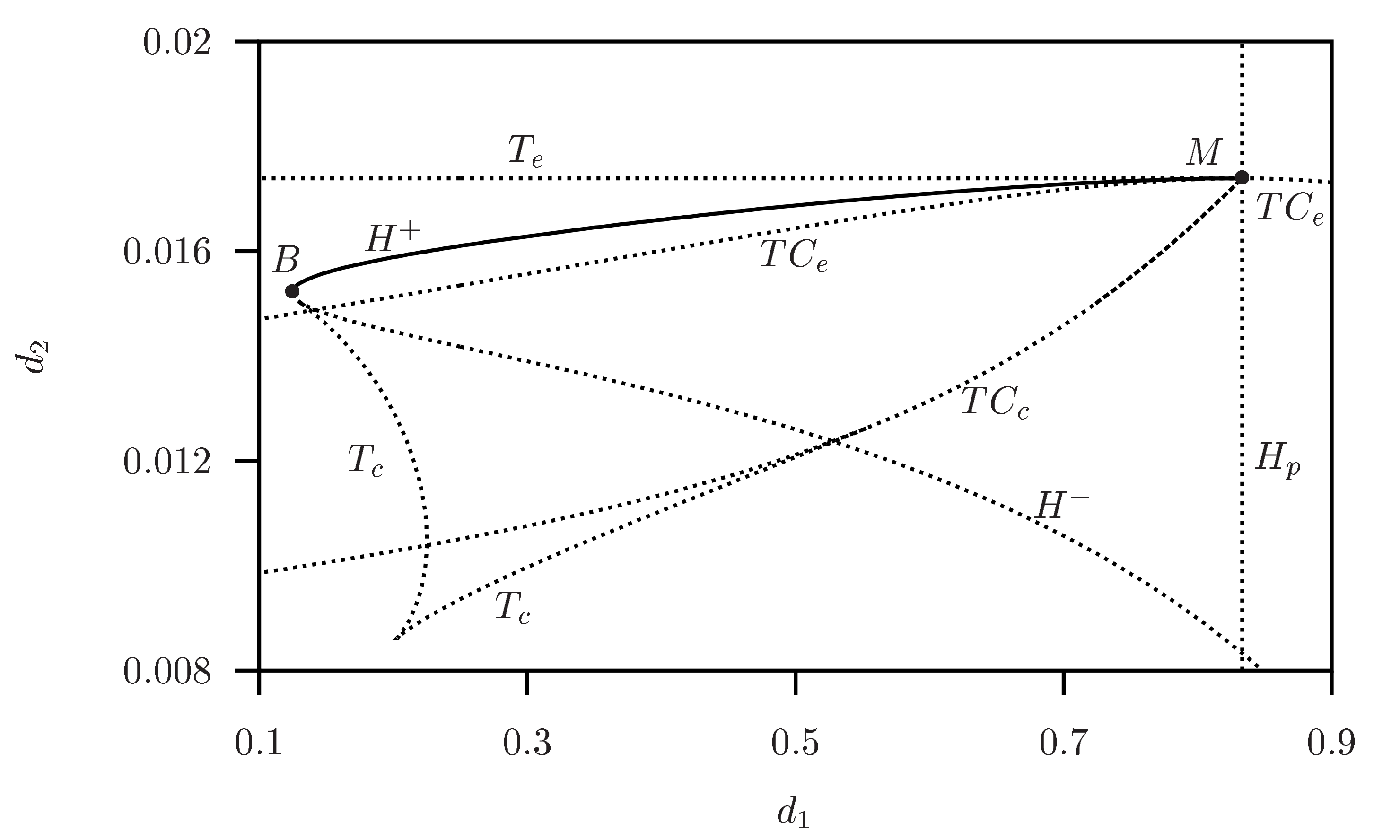
Local bifurcations



Math. Biosciences **134** (1996), 1-33

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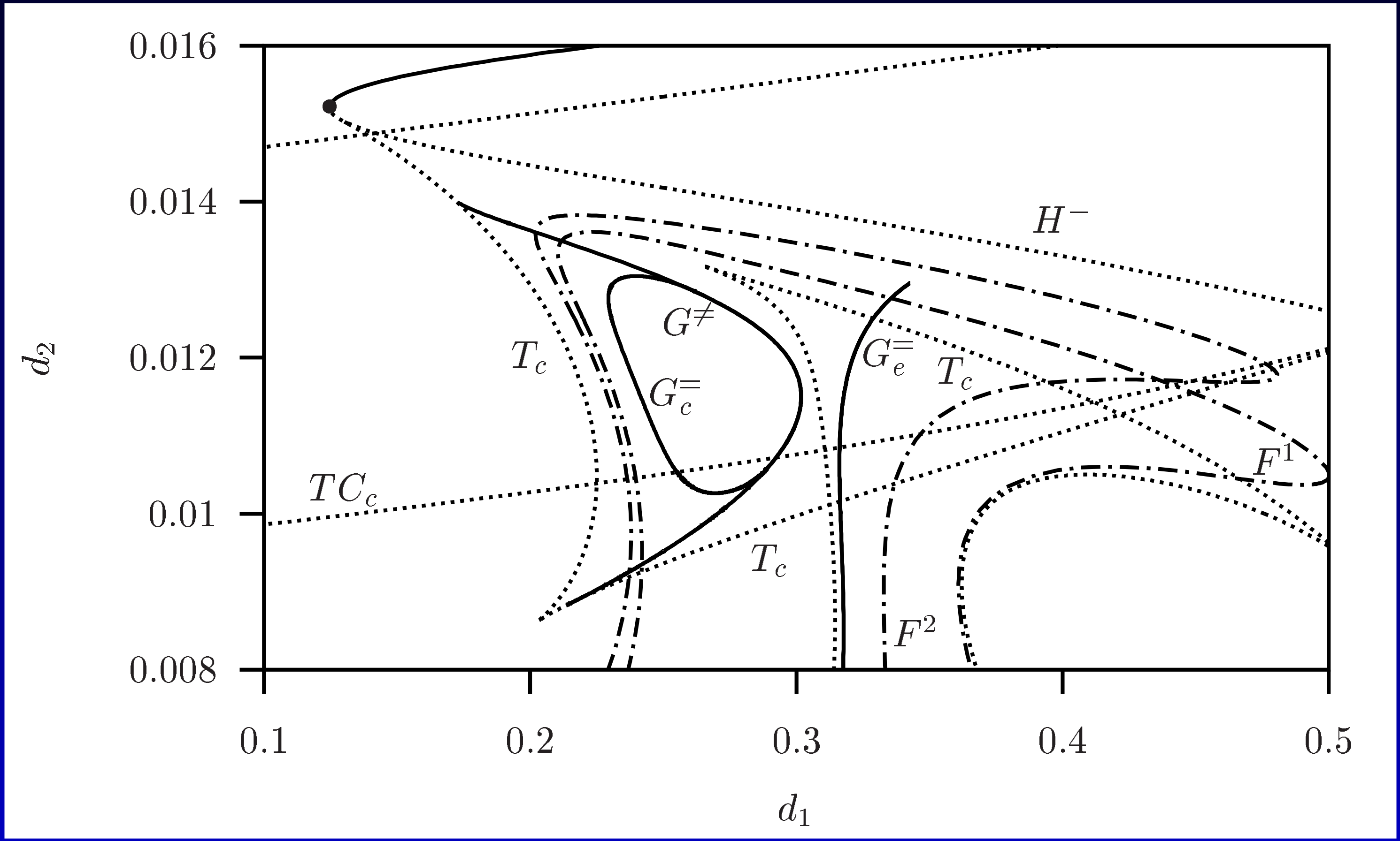
Local and key global bifurcations



Int. J. Bifurcation & Chaos **18** (2008), 1889-1903

Int. J. Bifurcation & Chaos **19** (2009), 159-169

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Trends

- Larger dimensions and codimensions;

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Trends

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- Semi-local and global phenomena in phase and parameter spaces;

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- Non-standard dynamical models (non-smooth, hybrid, constrained, spatially-distributted delays, etc.);

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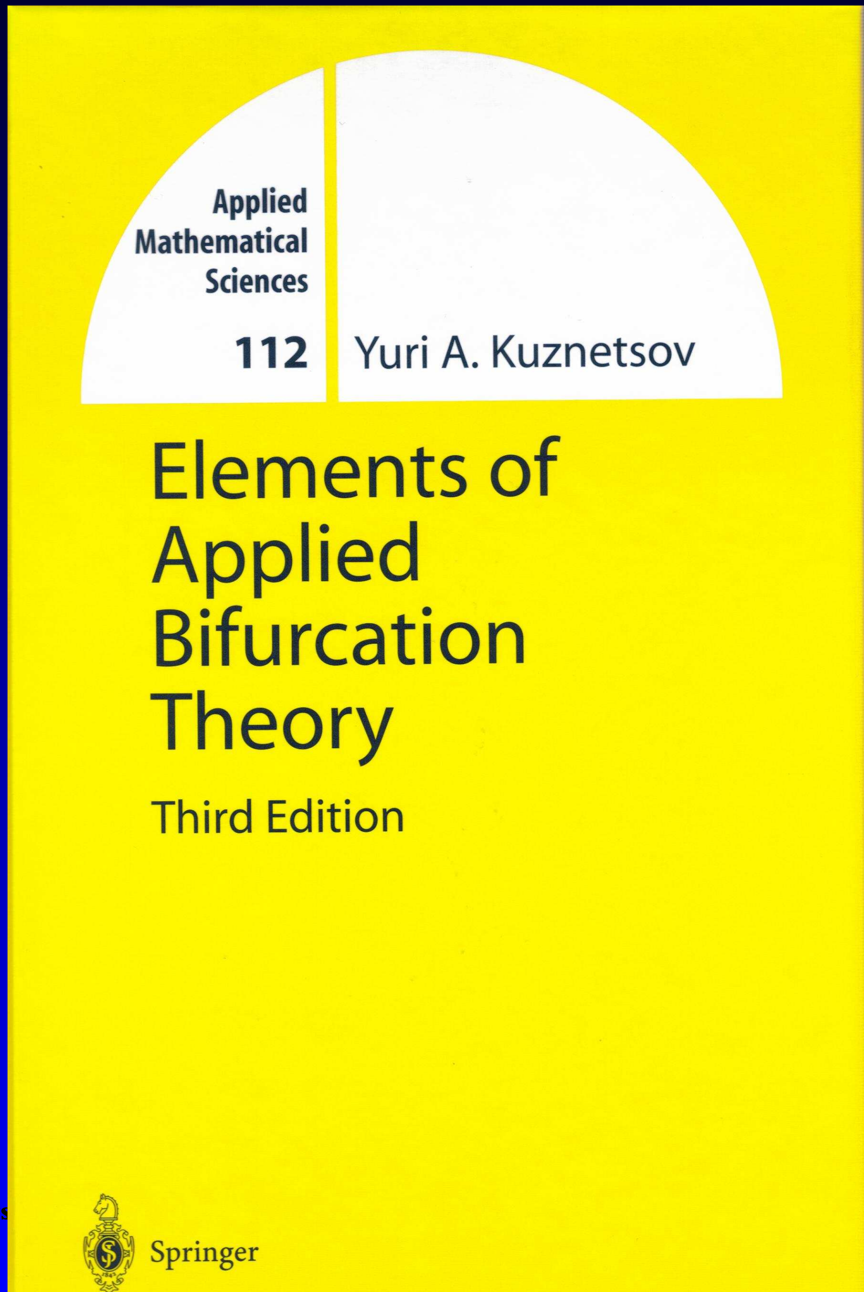
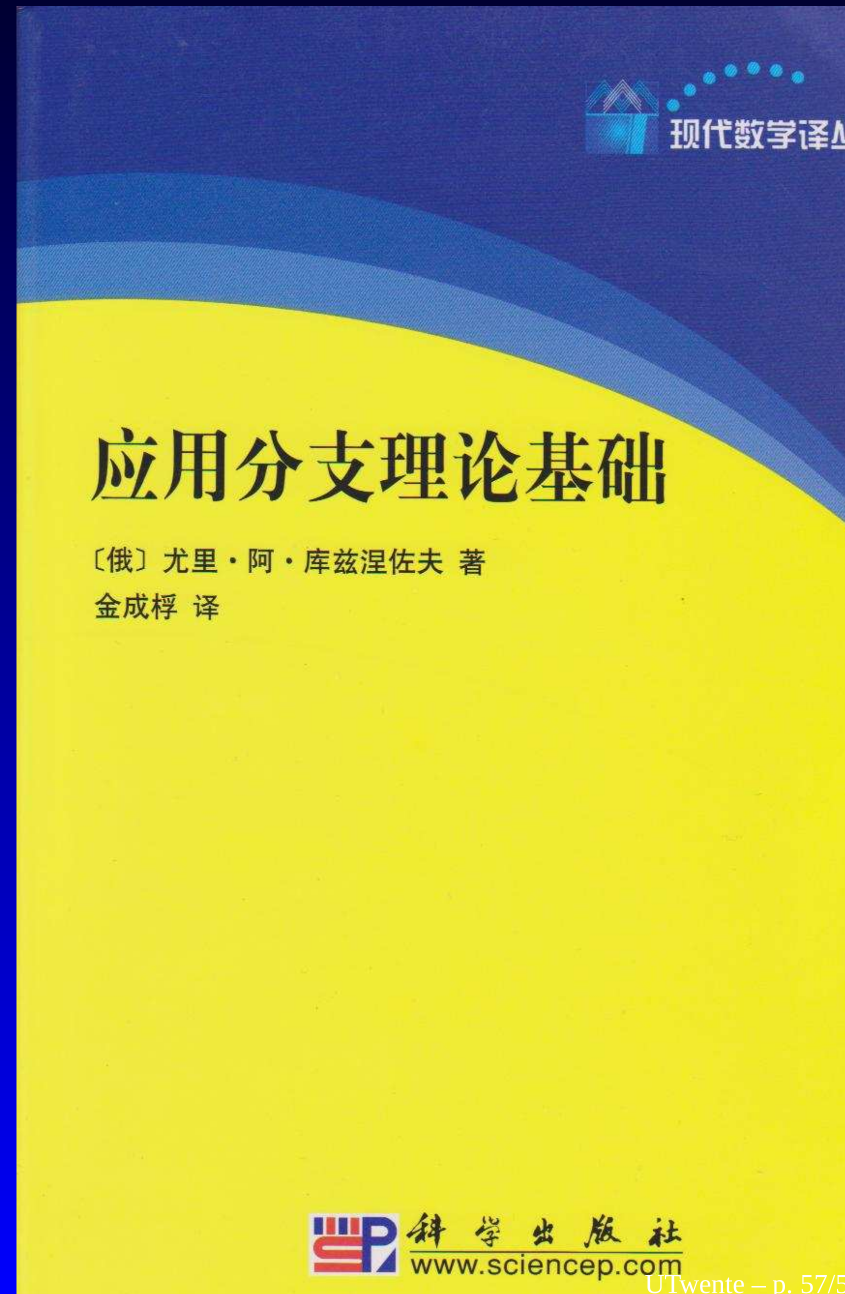
- Larger dimensions and codimensions;
- Semi-local and global phenomena in phase and parameter spaces;
- Non-standard dynamical models (non-smooth, hybrid, constrained, spatially-distributted delays, etc.);
- Implication of connection topology on network dynamics;

6. References





Book projects

- Yu.A. Kuznetsov, O. Diekmann, W.-J. Beyn. *Dynamical Systems Essentials*: An application oriented introduction to ideas, concepts, examples, methods, and results. Springer (polishing stage).

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- Yu.A. Kuznetsov and H.G.E. Meijer *Codimension Two Bifurcations of Iterated Maps*. Cambridge University Press.

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- Yu.A. Kuznetsov and H.G.E. Meijer *Codimension Two Bifurcations of Iterated Maps*. Cambridge University Press.
- A.R. Champneys and Yu.A. Kuznetsov. *Homoclinic Connections*: Localized phenomena in science and engineering. Springer.