The Fermat-Catalan equation in positive characteristic

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We consider the Fermat-Catalan equation

$$x^m + y^n + z^r = 0, \tag{1}$$

where m, n, r are positive integers such that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{r} < 1. \tag{2}$$

When x, y and z are taken to be non-zero integers satisfying $\gcd(x, y, z) = 1$, the Fermat-Catalan has been extensively studied in the literature. Despite these efforts there are still many open problems. For example it is still unknown if there are finitely many distinct triplets of values (x^m, y^n, z^r) or if there is any solution with $m, n, r > 2$.

In this article we analyze the function field analogue of the Fermat-Catalan equation and we will be especially interested in the characteristic p case. But first we consider the characteristic 0 case. So let k be an algebraically closed field of characteristic 0. Then, using Mason's Theorem, it follows that there are no non-constant solutions $x, y, z \in k[t]$ with $\gcd(x, y, z) = 1$, see for example section 5 of [1].

Now let k be an algebraically closed field of characteristic p. Then any solution of

$$x + y + z = 0$$

gives rise to solutions

$$x^{p^t} + y^{p^t} + z^{p^t} = 0$$

for all $t \in \mathbb{Z}_{\geq 0}$ due to Frobenius. So we need to impose the extra condition that $x^m, y^n, z^r \notin k[t^p]$. We are ready to formulate our Theorem.

Theorem 1. *Let k be an algebraically closed field of characteristic p. Then there are no non-constant solutions (x, y, z, m, n, r) of (1) satisfying $x, y, z \in k[t]$, $x^m, y^n, z^r \notin k[t^p]$, $\gcd(x, y, z) = 1$ and (2).*

Proof. Let us recall Mason's Theorem. We assume that the reader is familiar with the basic theory of function fields and heights. Let x_1, x_2, x_3 be non-zero elements of a function field K over an algebraically closed field k of characteristic p. Let S be a finite set of valuations such that

$$x_1 + x_2 + x_3 = 0$$

and

$$v(x_1) = v(x_2) = v(x_3)$$

for all $v \in M_K \setminus S$. Then either $x_1/x_2 \in K^p$ or

$$H_K(x_1/x_2) \leq 2g_{K/k} - 2 + |S|.$$

We use this with $K = k(t)$, $S = \{v : v(x_i) \neq 0 \text{ for some } i = 1, 2, 3\}$ and $(x_1, x_2, x_3) = (x^m, y^n, z^r)$. Because $K = k(t)$ is a rational function field, we know that $g_{K/k} = 0$. Furthermore,

$$\begin{aligned} |S| &= |\{v : v(x_i) \neq 0 \text{ for some } i = 1, 2, 3\}| \\ &= |\{v : v(x) \neq 0 \text{ or } v(y) \neq 0 \text{ or } v(z) \neq 0\}| \\ &\leq \deg(x) + \deg(y) + \deg(z) + 1, \end{aligned}$$

where the extra 1 accounts for the infinite valuation.

Now assume that $x^m/y^n \notin K^p$. Then Mason's Theorem tells us that

$$\max\{m \deg x, n \deg y\} = H_K(x^m/y^n) < \deg(x) + \deg(y) + \deg(z).$$

Note that our assumption $x^m/y^n \notin K^p$ implies $x^m/z^r \notin K^p$. So in exactly the same way we find that

$$\max\{m \deg x, r \deg z\} = H_K(x^m/z^r) < \deg(x) + \deg(y) + \deg(z).$$

We conclude that

$$\max\{m \deg x, n \deg y, r \deg z\} < \deg(x) + \deg(y) + \deg(z),$$

but this is impossible due to our assumption (2).

We still need to consider the case $x^m/y^n \in K^p$. Observe that then also $x^m/z^r \in K^p$. Recall that $x^m \notin k[t^p]$, hence we can take a valuation $v \neq \infty$ such that $p \nmid mv(x)$. On the other hand $x^m/y^n \in K^p$ and $x^m/z^r \in K^p$ shows that

$$\begin{aligned} p &\mid mv(x) - nv(y) \\ p &\mid mv(x) - rv(z). \end{aligned}$$

Hence

$$mv(x) \equiv nv(y) \equiv rv(z) \not\equiv 0 \mod p.$$

This contradicts our assumption $\gcd(x, y, z) = 1$. □

References

[1] H. Darmon and A. Granville, *On the equations $z^m = F(x, y)$ and $Ax^p + By^q = Cz^r$*, Bull. London Math. Soc., 27 (1995), pp. 513–543.