# Vinogradov's method and class groups

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Bonn 2018

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# Vinogradov's method

**Theorem 1 (Vinogradov's theorem, 1930s)**

*Every sufficiently large odd integer can be written as the sum of three prime numbers.*

Need to estimate

$$\sum_{p \leq X} e^{2\pi i \alpha p}$$

for $\alpha \in \mathbb{R}$.

Vinogradov's idea: a new method to deal with sums over primes.

Friedlander and Iwaniec expanded on the ideas of Vinogradov, and used them to prove their famous result that $X^2 + Y^4$ is prime infinitely often.

# Class groups

Let $p$ be a prime number and let $h(-p)$ be the class number of $\mathbb{Q}(\sqrt{-p})$.

By Gauss genus theory, the 2-part of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p}))$ is cyclic.

Hence the 2-adic valuation of $h(-p)$ determines the group structure of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p}))[2^\infty]$. We have

$$2 \mid h(-p) \Leftrightarrow p \text{ splits completely in } \mathbb{Q}(i)$$
$$4 \mid h(-p) \Leftrightarrow p \text{ splits completely in } \mathbb{Q}(\zeta_8)$$
$$8 \mid h(-p) \Leftrightarrow p \text{ splits completely in } \mathbb{Q}(\zeta_8, \sqrt{1+i}).$$

This allows one to use the Chebotarev density theorem to prove density results for the divisibility of $h(-p)$ by $2^k$ for $k = 1, 2, 3$.

No such field is known for $16 \mid h(-p)$.

# Density result

Observation due to Milovic: criteria in the literature for $16 \mid h(-p)$ are very similar to the symbols appearing in the work of Friedlander and Iwaniec on $X^2 + Y^4$.

Use Vinogradov's method!

**Theorem 2 (K.)**

*The natural density of prime numbers $p$ for which 16 divides $h(-p)$ is $\frac{1}{16}$.*

This is an improvement of earlier work of Koymans and Milovic, who proved the same result under the assumption of a short character sum conjecture.