An overview of arithmetic statistics

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Techniques from arithmetic statistics can also be used to prove results that do not have a statistical nature.

Hilbert's tenth problem

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In modern terms: does there exist an algorithm such that:
Input: a polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$.
Output: "YES" if there is an integer solution $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $p(a_1, \ldots, a_n) = 0$, "NO" otherwise.

Finitely generated rings

Theorem (MRDP, 1970)

Hilbert's tenth problem is undecidable, i.e. there is no algorithm that can decide whether a polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$ has a zero or not.

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In the process, we also prove a conjecture of Denef–Lipshitz (1978).

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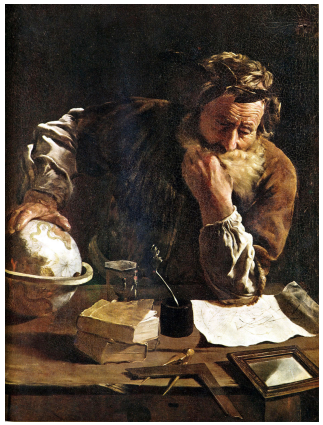
Solution x, y	Ratio x/y	Expansion of $\sqrt{2}$
$x = 1,\ y = 1$	1	1.4142135...
$x = 3,\ y = 2$	1.5	1.4142135...
$x = 7,\ y = 5$	1.4	1.4142135...
$x = 17,\ y = 12$	1.4166666...	1.4142135...
$x = 41,\ y = 29$	1.4137931...	1.4142135...
$x = 99,\ y = 70$	1.4142857...	1.4142135...

The positive Pell equation



Archimedes

Dirichlet proved that one can always non-trivially solve

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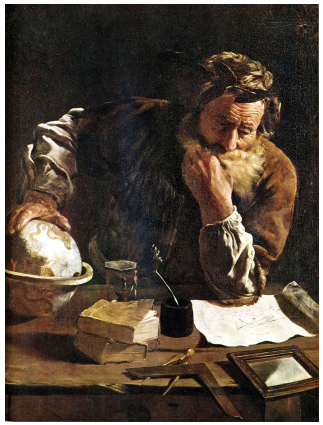

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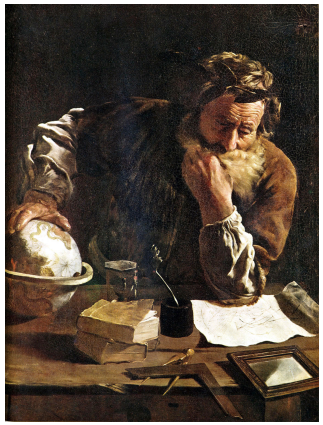

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Example (Fermat's challenge to Brouncker and Wallis)

The smallest non-trivial solution to $x^2 - 61y^2 = 1$ *is*

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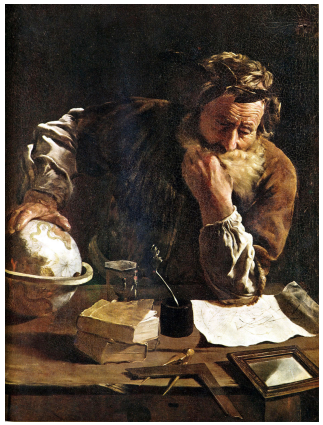

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The smallest non-trivial solution to $x^2 - 61y^2 = 1$ *is*

$$x = 1766319049, \quad y = 226153980.$$

The negative Pell equation

To solve the negative Pell equation $x^2 - dy^2 = -1$, one certainly needs to be able to solve it modulo p for all primes p. But if p divides d, we get

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Nagell (1930s) conjectured

$$\lim_{X \to \infty} \frac{\#\{d \leq X : d \in \mathcal{D}, x^2 - dy^2 = -1 \text{ sol.}\}}{\#\{d \leq X : d \in \mathcal{D}\}}$$

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Theorem (K.–Pagano, 2022)

Nagell's and Stevenhagen's conjecture are true.

Sums of cubes

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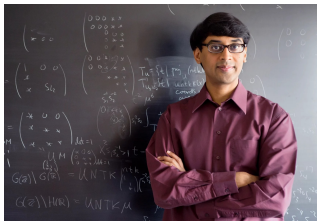
Example (Rational versus integer solutions)

For $n = 6$ one can use the factorization $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ to show that there are no integer solutions. However, we have

$$6 = \left(\frac{17}{21}\right)^3 + \left(\frac{37}{21}\right)^3.$$

Results

Levent Alpöge



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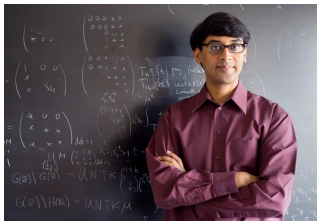

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At least 31.4% *of the integers are not sums of two rational cubes. Assuming a parity conjecture, at least* 47.4% *of integers are sums of two rational cubes.*

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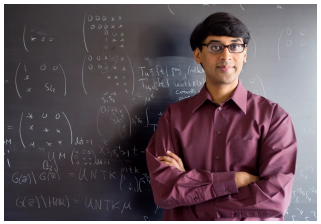

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Conjecturally, the limit should be $1/2$.

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The rational points of an elliptic curve are

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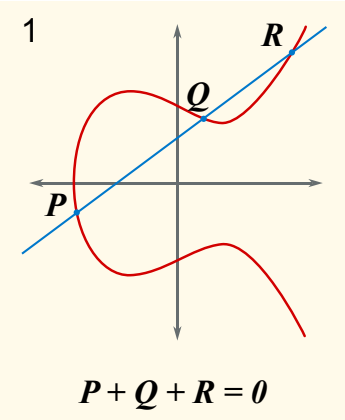
An amazing fact is that $E(\mathbb{Q})$ is naturally a finitely generated abelian group.



The group law on $E(\mathbb{Q})$

Connecting elliptic curves to sums of rational cubes

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In fact, since $C_n \cong C_1$ over $\mathbb{Q}(\sqrt[3]{n})$, the family C_n is called a "cubic twist family".

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Our construction fixes some $a_1, a_2, a_3 \in K$ and then considers the family

$$E^d : dy^2 = (x - a_1)(x - a_2)(x - a_3)$$

for varying d. Note that $E^d \cong E^1$ over $K(\sqrt{d})$ (in particular over $\overline{\mathbb{Q}}$) by changing y to $y/\sqrt{d}$. This is known as a *quadratic twist family*.

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Once more: all of the curves C_d are isomorphic over $\overline{\mathbb{Q}}$.

Generalized class groups

Class groups go back to Gauss: these measure how close rings such as $\mathbb{Z}[\zeta_n]$, $\mathbb{Z}[\sqrt{-3}]$, $\mathbb{Z}[\sqrt{-105}]$ and $\mathbb{Z}[\sqrt{d}]$ are to having "unique factorization".

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There is a "generalized class group" (known as Selmer group) that gives an upper bound on how many independent rational points an elliptic curve has.

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- and for Hilbert's tenth problem it is $H^1(G_K, E^d[2^\infty])$.

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Since the relevant curves are isomorphic over $\overline{\mathbb{Q}}$, this gives the extra structure:

$$\mathbb{F}_2(\chi_d) \cong \mathbb{F}_2(\chi_{d'}) = \mathbb{F}_2, \qquad C_n[3] \cong C_{n'}[3], \qquad E^d[2] \cong E^{d'}[2].$$

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$$\begin{cases} 0.5 & \text{if } r = 0 \\ 0.5 & \text{if } r = 1 \\ 0 & \text{if } r \geq 2. \end{cases}$$

High-level proof overview

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Step 2:

Analysis: Equidistribution of comparison function

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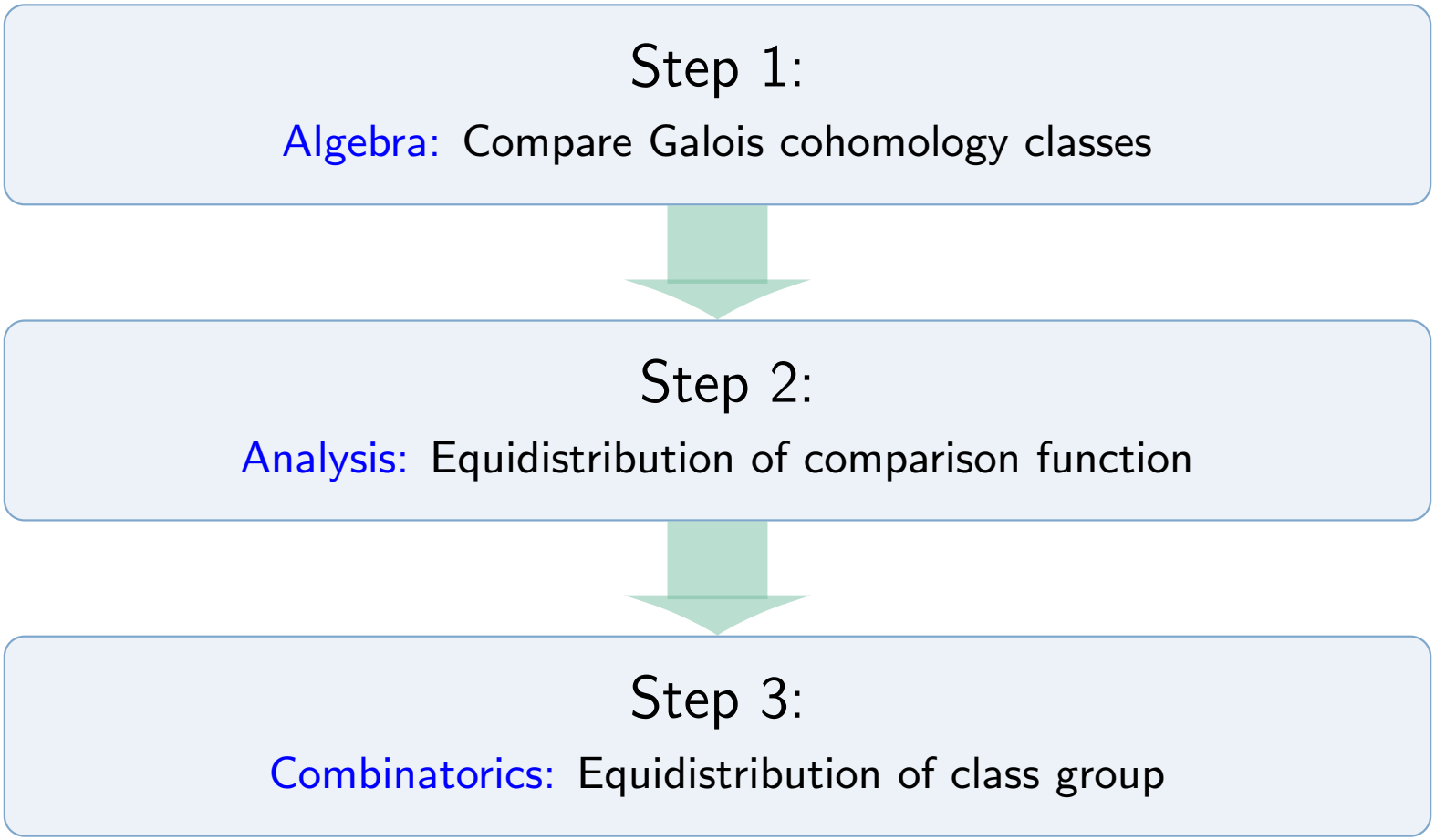
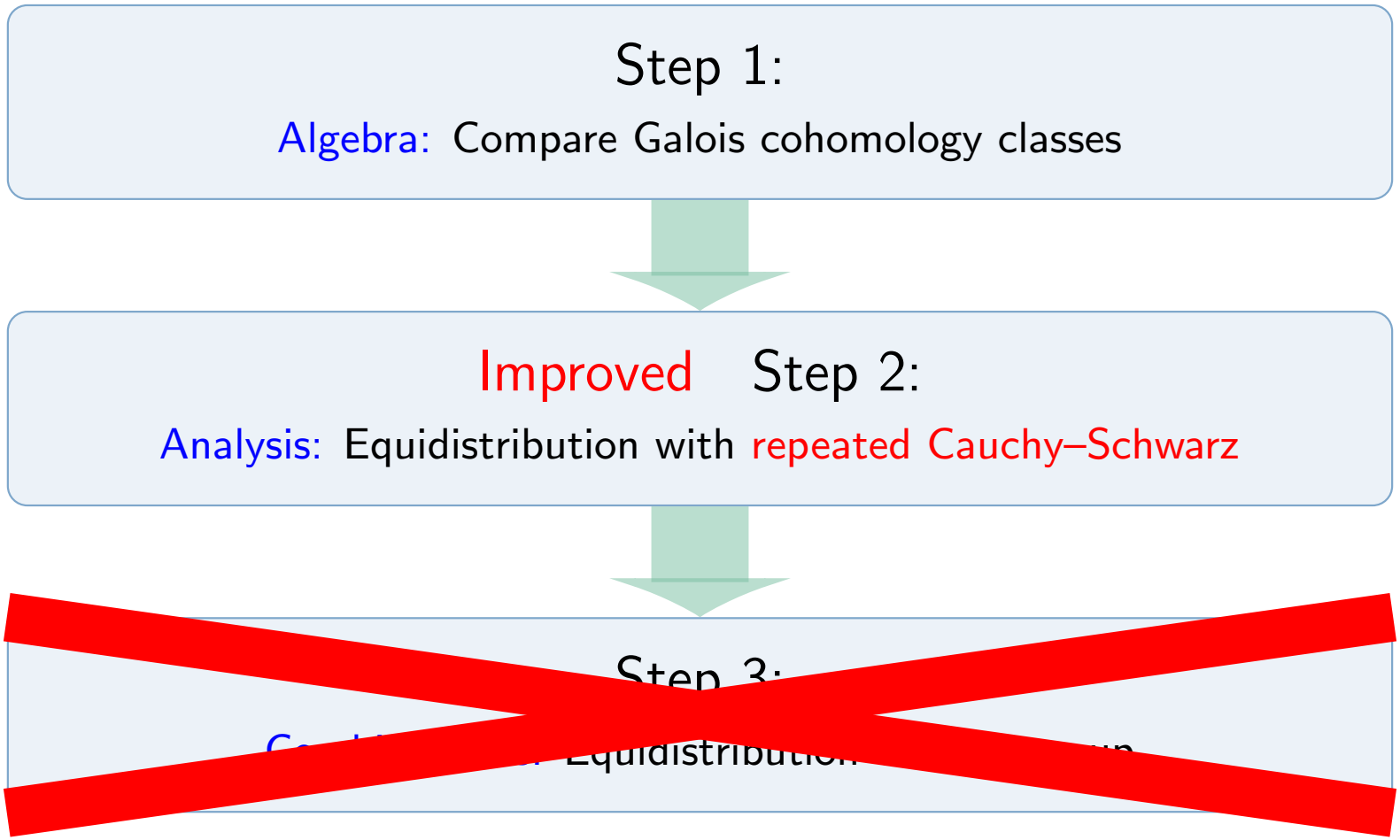


High-level proof overview

Lecture series:



Repeated Cauchy–Schwarz

Rough idea: given a product space, say $X := X_1 \times X_2 \times X_3$ with each X_i a subset of the prime numbers, we want to prove that

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$$\sum_{(p,q,r)\in X} g(pqr) \leq \sum_{(q,r)\in X_2\times X_3} 1 \cdot \left| \sum_{p\in X_1} g(pqr) \right|$$

$$\leq |X_2 \times X_3|^{1/2} \left(\sum_{p_1,p_2\in X_1} \sum_{(q,r)\in X_2\times X_3} g(p_1 qr) g(p_2 qr) \right)^{1/2}.$$

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Doing this once over every interval X_i, one gets exactly the sum

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Now one can directly use our Step 1 to replace $\prod_{i,j,k\in\{1,2\}} g(p_i q_j r_k)$ by a symbol for which we can prove equidistribution, thus giving equidistribution of $g(x)$ on X.

Questions?

Thank you for your attention! Quick recap:

1. Introduction
2. Hilbert's tenth problem
3. The negative Pell equation
4. Sums of rational cubes
5. The common theme
6. High-level proof overview