Factoring in number rings

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Basic arithmetic

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But what happens for other rings? Many applications ranging from abstract number theory and geometry to cryptography.

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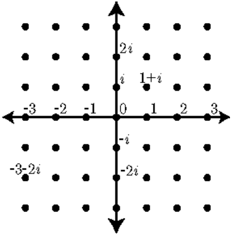

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$$(1 + i) + (-3 - 2i) = -2 - i$$

and multiply Gaussian integers by expanding brackets and using the rule $i^2 = -1$

$$(2 + 5i) \cdot (3 - 4i) = 6 + 15i - 8i - 20i^2 = 26 + 7i.$$

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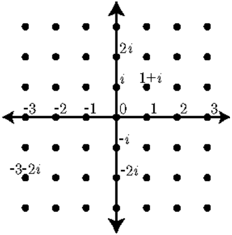

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Every Gaussian integer can uniquely be factored as a unit and Gaussian primes. The units of $\mathbb{Z}[i]$ are $\{\pm 1, \pm i\}$.

Primes and irreducibles

To study factorization in more general rings, we make a definition.

Definition 1

Let R be a commutative ring. We say that $a \in R \setminus \{0\}$ is irreducible if it is not the product of two non-units. Furthermore, an element $a \in R$, that is non-zero and not a unit, is called prime if for all $b, c \in R$ we have $a \mid bc$ implies $a \mid b$ or $a \mid c$.

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If R is an integral domain, then every prime element is irreducible. The converse also holds for $R = \mathbb{Z}$ and $R = \mathbb{Z}[i]$, but it does not hold in general.

Failure of unique factorization

In the ring $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$ we have

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

It is not hard to see that 2, 3, $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are all irreducible. The problem comes from the fact that 3 is irreducible but not prime.

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Theorem 1

If R is a a number ring, then every element of R can be factored into irreducible elements.

Kummer and ideals

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This led to the introduction of an abelian group called the class group, which measures the failure of unique factorization of elements.

Class groups are known to be finite, but still very mysterious. Cohen and Lenstra conjectured that class groups behave like random finite abelian groups in families of number fields.

Random finite abelian groups

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a	$*$	$*$	$*$	$*$
b	$*$	$*$	$*$	$*$
c	$*$	$*$	$*$	$*$
d	$*$	$*$	$*$	$*$

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Then we get the finite abelian groups of size 4 with probability proportional to $\frac{1}{|\mathrm{Aut}(A)|}$.

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Theorem 2 (K., 2018)

The density of prime numbers p such that the class number of $\mathbb{Q}(\sqrt{-p})$ *is divisible by* 16 *is equal to* $\frac{1}{16}$.

Current research: wide families

Let l be an odd prime number. If K is a degree l cyclic field, then $\mathrm{Cl}(K)$ becomes a $\mathbb{Z}[\zeta_l]$-module in a natural way. The following theorem generalizes a result due to Smith.

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Theorem 3 (K.-Pagano, 2018)

Assume GRH and let l be an odd prime number. Then the group $((1 - \zeta_l)\mathrm{Cl}(K))[l^\infty]$ has the distribution predicted by Cohen and Lenstra as K varies over degree l cyclic fields over $\mathbb{Q}$ ordered by discriminant.

Questions