Malle's conjecture for nonic Heisenberg extensions

Peter Koymans
Max Planck Institute for Mathematics

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Conjecture 1 (Inverse Galois problem)

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A famous theorem due to Shafarevich (1954) shows that the answer is yes for G solvable.

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Conjecture 2 (Malle's conjecture)

There are $a(G), c(G) > 0$ and $b(G) \in \mathbb{Z}_{>0}$ such that

$$N(G,X) \sim c(G) X^{a(G)} (\log X)^{b(G)-1}.$$

Malle gave explicit values for $a(G)$ and $b(G)$ but NOT for $c(G)$.

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The value of $a(G)$ is generally believed to be correct. The value of $c(G)$ is sometimes given by an infinite product over primes p, where the factors are certain local densities (Malle–Bhargava principle).

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- direct products $S_n \times A$ for $n \in \{3, 4, 5\}$ and A abelian by Wang (with $\#A$ coprime to some values) and later by Masri–Thorne–Tsai–Wang.

The weak form of Malle's conjecture

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The weak form is known for nilpotent G by Klüners–Malle with further progress in the solvable case by Alberts and Alberts–O'Dorney.

The Heisenberg group

Let ℓ be a prime number and let Heis_ℓ be the multiplicative group

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For $\ell = 2$ we get $\mathrm{Heis}_2 \cong D_4$.

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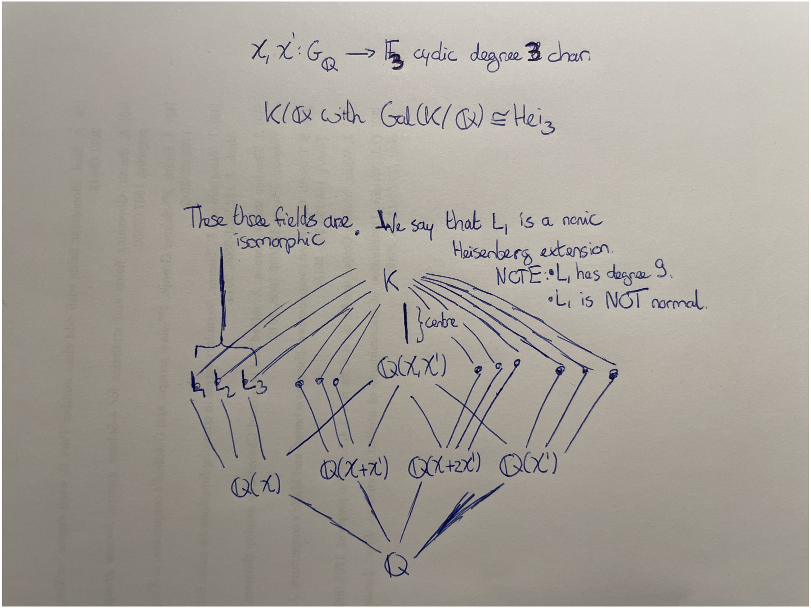
We give an explicit value for the constant c.

Subfield diagram of Heis$_3$



Comparison with quartic D_4 extensions

Cohen–Diaz y Diaz–Olivier proved that

$$N(\mathrm{Heis}_2, X) \sim \frac{6X}{\pi^2} \sum_{D} \frac{2^{-i(D)}}{D^2} \frac{L(1, D)}{L(2, D)},$$

where the sum is over fundamental quadratic discriminants and $i(D) = 0$ if $D > 0$ and $i(D) = 1$ if $D < 0$.

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- count, uniformly in the number field K, the number of quadratic extensions of K with relative discriminant bounded by X;
- sum this function over all quadratic number fields K.

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This fails for cyclic degree 3 extensions. Need a new strategy!

The strategy

Step 1: given two linearly independent characters $\chi, \chi' : G_{\mathbb{Q}} \to \mathbb{F}_3$, there exists an Heisenberg extension containing $\mathbb{Q}(\chi, \chi')$ if and only if all ramified primes (not equal to 3) have residue field degree 1 in $\mathbb{Q}(\chi, \chi')$.

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A similar strategy was used by Heath-Brown to find the distribution of the 2-Selmer groups $\mathrm{Sel}^2(E^d)$ of quadratic twists d of an elliptic curve E, and by Fouvry–Klüners to find the distribution of $2\mathrm{Cl}(K)[4]$.

Lifting bicyclic extensions, I

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Writing $\psi = (\phi, \rho)$ with $\phi : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ any continuous map, we see that ψ is a homomorphism if and only if

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Hence a homomorphism ψ exists if and only if θ is trivial when inflated to $H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell)$, where we view $\mathbb{F}_\ell$ as a discrete $G_{\mathbb{Q}}$-module with trivial action.

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By class field theory we know that θ is trivial in $H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell)$ if and only if it is trivial in $H^2(G_{\mathbb{Q}_v}, \mathbb{F}_\ell)$ for every place v.

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Theorem 2 (Michailov)

Let ℓ be an odd prime number. Let $\chi, \chi' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ be two linearly independent characters. Then there exists a Heisenberg extension $M/\mathbb{Q}$ containing $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\chi')$ if and only if every ramified prime (not equal to ℓ) has residue field degree 1 in the bicyclic field $\mathbb{Q}(\chi, \chi')$.

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If such an extension $M/\mathbb{Q}$ exists, there are infinitely many, which can all be obtained by twisting M by a cyclic degree ℓ character of $G_{\mathbb{Q}}$.

Minimal Heisenberg extensions

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This is great, because the discriminant of a minimal Heisenberg extension is easily computed.

The discriminant of nonic Heisenberg extensions

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Lemma 4 (Fouvry–K.)

Let ℓ be an odd prime. Let M be a minimal Heisenberg extension containing $\mathbb{Q}(\chi, \chi')$ defined by a character ρ. Then up to factors of ℓ

$$\Delta_{M/\mathbb{Q}} = \Delta(\chi)^{\ell^2(\ell-1)} \mathrm{free}(\Delta(\chi'), \Delta(\chi))^{\ell^2(\ell-1)} = \prod_{p \mid \Delta(\chi)\Delta(\chi')} p^{\ell^2(\ell-1)}.$$

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Now twist ρ by a character $\chi'' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ ramified precisely at the primes dividing d, coprime with $\Delta(\chi)\Delta(\chi')$. Then up to factors of ℓ

$$\Delta_{\mathbb{Q}(\chi,\chi')(\rho+\chi'')/\mathbb{Q}} = d^{\ell^2(\ell-1)} \Delta(\chi)^{\ell^2(\ell-1)} \mathrm{free}(\Delta(\chi'), \Delta(\chi))^{\ell^2(\ell-1)}.$$

The discriminant of nonic Heisenberg extensions

For $\chi : G_{\mathbb{Q}} \to \mathbb{F}_\ell$, define $\Delta(\chi)$ to be the product of the ramified primes in $\mathbb{Q}(\chi)$. Define $\mathrm{free}(d, a)$ be the largest divisor of d coprime to a.

Lemma 4 (Fouvry–K.)

Let ℓ be an odd prime. Let M be a minimal Heisenberg extension containing $\mathbb{Q}(\chi, \chi')$ defined by a character ρ. Then up to factors of ℓ

$$\Delta_{M/\mathbb{Q}} = \Delta(\chi)^{\ell^2(\ell-1)} \mathrm{free}(\Delta(\chi'), \Delta(\chi))^{\ell^2(\ell-1)} = \prod_{p \mid \Delta(\chi)\Delta(\chi')} p^{\ell^2(\ell-1)}.$$

Now twist ρ by a character $\chi'' : G_{\mathbb{Q}} \to \mathbb{F}_\ell$ ramified precisely at the primes dividing d, coprime with $\Delta(\chi)\Delta(\chi')$. Then up to factors of ℓ

$$\Delta_{\mathbb{Q}(\chi,\chi')(\rho+\chi'')/\mathbb{Q}} = d^{\ell^2(\ell-1)} \Delta(\chi)^{\ell^2(\ell-1)} \mathrm{free}(\Delta(\chi'), \Delta(\chi))^{\ell^2(\ell-1)}.$$

Let $\mathbb{Q}(\chi) \subsetneq L \subsetneq \mathbb{Q}(\chi, \chi')(\rho + \chi'')$. Then up to factors of ℓ

$$\Delta_{L/\mathbb{Q}} = d^{\ell(\ell-1)} \Delta(\chi)^{\ell(\ell-1)} \mathrm{free}(\Delta(\chi'), \Delta(\chi))^{(\ell-1)^2}.$$

Wild ramification

To find the exponent of ℓ in the discriminant, recall that $\mathbb{Q}_\ell^*/\mathbb{Q}_\ell^{*\ell}$ is of dimension 2.

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Hence there are 2 linearly independent characters $G_{\mathbb{Q}_\ell} \to \mathbb{F}_\ell$ of which the discriminant is easily computed.

Theorem 5 (Fouvry–K.)

Let ℓ be an odd prime. Then there exists precisely one Heisenberg extension $M/\mathbb{Q}_\ell$. Its discriminant ideal equals

$$(\ell)^{\ell(\ell+1)(2\ell-2)}.$$

The number of nonic Heisenberg extensions

We have now parametrized nonic Heisenberg extensions and we have computed their discriminant.

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Theorem 6 (Fouvry–K.)

Let ℓ be an odd prime number. Then

$$N(\mathrm{Heis}_\ell, X) = \frac{1}{\ell^3(\ell-1)^2} \sum_{\substack{\chi,\chi' : G_{\mathbb{Q}} \to \mathbb{F}_\ell \\ \chi,\chi' \text{ linearly independent}}} \mathbf{1}_{\theta_{\chi,\chi'}(\sigma,\tau) \text{ trivial}} \cdot \ell^{\omega(\Delta(\chi)\Delta(\chi'))} \cdot T(X, \chi, \chi', \ell),$$

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where

$$T(X, \chi, \chi', \ell) = \sum_{\substack{d \in \mathbb{Z}_{>0} \\ \gcd(d, \Delta(\chi)\Delta(\chi'))=1 \\ p \mid d \Rightarrow p \equiv 0,1 \bmod \ell \\ d^{\ell(\ell-1)} \leq \frac{X}{\Delta(\chi)^{\ell(\ell-1)} \mathrm{free}(\Delta(\chi'),\Delta(\chi))^{(\ell-1)^2} \mu(\chi,\chi',d)}}} \mu^2(d) \cdot (\ell-1)^{\omega(d)}.$$

A character sum

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Viewing χ_1, χ_2 as Dirichlet characters of order ℓ (so taking values in $\mathbb{C}^*$)

$$\mathbf{1}_{\theta_{\chi_1,\chi_2}(\sigma,\tau)\text{ trivial}} = \prod_{\substack{r \mid \Delta(\chi_1)\Delta(\chi_2) \\ r \neq \ell}} \frac{1}{\ell} \left(\sum_{\substack{(z_1,z_2) \in \mathbb{F}_\ell^2 \\ \chi_1^{z_1}\chi_2^{z_2} \text{ unr. at } r}} (\chi_1^{z_1}\chi_2^{z_2})(r) \right).$$

Evaluating the character sum for $\ell = 3$

We restrict the sum to pairs (χ, χ') with $\Delta(\chi)$ small (i.e. smaller than a power of $\log X$), and with $\Delta(\chi')$ large (i.e. close to $X^{1/4}$).

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The main term comes from the r dividing $\Delta(\chi')$ and not dividing $\Delta(\chi)$. Indeed, the prime r contributes the following

$$1 + \chi(r) + \chi^2(r)$$

in the above product for $\mathbf{1}_{\theta_{\chi,\chi'}(\sigma,\tau) \text{ trivial}}$.

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Since χ' has huge conductor and r is small, we get oscillation when summing over χ'. This follows from the Siegel–Walfisz theorem.

What about $\ell > 3$?

The algebraic results are valid for all odd primes ℓ, but we use that $\ell = 3$ when evaluating the character sum for the following reasons:

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It is easy to extend our results to any odd prime ℓ for which $\mathbb{Z}[\zeta_\ell]$ is a PID (i.e. $\ell \in \{3, 5, 7, 11, 13, 17, 19\}$).

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It is plausible that our results can also be extended to any odd prime ℓ.

That's it!

Thank you for your attention!