Malle's conjecture for nonic Heisenberg extensions, pre-talk

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MAX-PLANCK-GESELLSCHAFT

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Overview

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- The statement of Malle's conjecture;
- The structure of the Heisenberg group;
- Central extensions and some basic Galois cohomology.

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This subgroup is only well-defined up to conjugation: relabelling the embeddings gives a conjugate subgroup.

Statement of Malle's conjecture

For a transitive subgroup $G \subseteq S_n$, consider the counting function

$$N(G, X) := \#\{K/\mathbb{Q} : [K : \mathbb{Q}] = n, \mathrm{Gal}(K/\mathbb{Q}) \cong_{\text{perm. gr.}} G, \Delta_{K/\mathbb{Q}} \leq X\},$$

where $\Delta_{K/\mathbb{Q}}$ denotes the discriminant and K is taken inside a fixed algebraic closure $\overline{\mathbb{Q}}$.

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Conjecture 1 (Malle's conjecture)

There exists a constant $c(G) > 0$ such that

$$N(G, X) \sim c(G) X^{a(G)} (\log X)^{b(G)-1}.$$

Malle gave explicit values for the constants $a(G)$ and $b(G)$.

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Furthermore, Malle proposed

$$b(G) := \#\{C \in \mathrm{Conj}(G) : \mathrm{ind}(C) = a(G)^{-1}\}/\sim,$$

where two conjugacy classes C and C' are equivalent if they are in the same orbit under a certain action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Conj}(G)$.

The Heisenberg group

Let ℓ be an odd prime number. The Heisenberg group Heis_ℓ is the multiplicative group of upper triangular matrices with ones on the diagonal and entries in $\mathbb{F}_\ell$:

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Theorem 1 (Basic facts about Heis_ℓ)

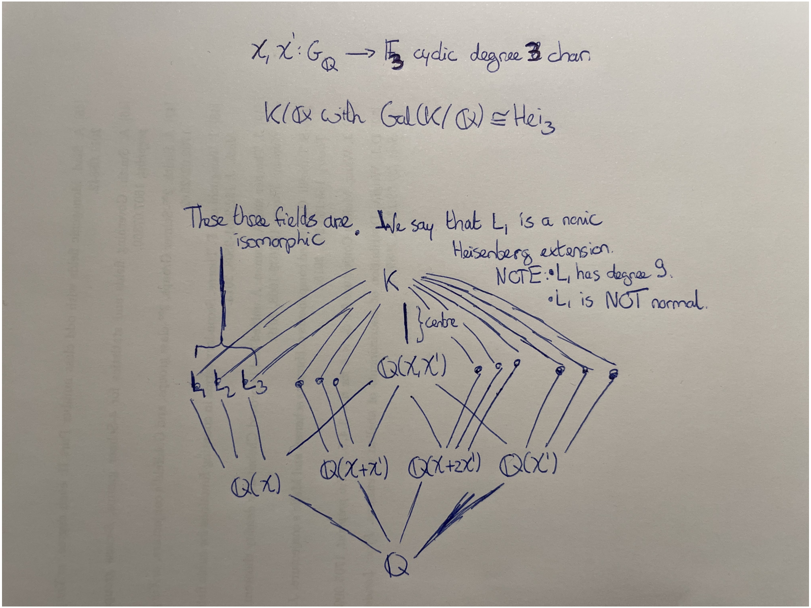
Let ℓ be an odd prime. We have the following

- *every element of* Heis_ℓ *has order* ℓ*;*
- *the centre* $Z(\mathrm{Heis}_\ell)$ *of* Heis_ℓ *is of size* ℓ*;*
- *there is an exact sequence*

$$1 \to \mathbb{F}_\ell \to \mathrm{Heis}_\ell \to \mathbb{F}_\ell^2 \to 1$$

with the image of $\mathbb{F}_\ell$ *landing in* $Z(\mathrm{Heis}_\ell)$*.*

Subfield diagram of Heis_3



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Fortunately, Heis_3 is a very symmetric group so all 12 subgroups H of order 3 lead to the same conjugacy class of subgroups in S_9. Concretely, generators are

$$(1,2,9)(3,4,5)(6,7,8),\ (3,4,5)(6,8,7),\ (1,4,7)(2,5,8)(3,6,9),$$

so $a(\mathrm{Heis}_3) = 4$, $b(\mathrm{Heis}_3) = 1$.

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so $a(\mathrm{Heis}_3) = 4$, $b(\mathrm{Heis}_3) = 1$. Hence, conjecturally, there is $c > 0$ with

$$N(\mathrm{Heis}_3, X) \sim cX^{1/4}.$$

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We will develop some general tools to answer the following question:

when does there exist a normal, degree 27 extension $K/\mathbb{Q}$ containing $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\chi')$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Heis}_3$?

Central extensions

Recall that we had an exact sequence

$$1 \to \mathbb{F}_\ell \to \mathsf{Heis}_\ell \to \mathbb{F}_\ell^2 \to 1$$

with the image of $\mathbb{F}_\ell$ landing in $Z(\mathsf{Heis}_\ell)$.

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Theorem 2 (Heis$_\ell$ as a central extension)

Let $\chi_1, \chi_2 : \mathbb{F}_\ell^2 \to \mathbb{F}_\ell$ be the natural projections. Then the Heisenberg extensions (i.e. those with $E \cong \mathrm{Heis}_\ell$) correspond to the non-trivial multiples of $(v, w) \mapsto \chi_1(v) \cdot \chi_2(w)$ inside $H^2(\mathbb{F}_\ell^2, \mathbb{F}_\ell)$.

Inflation–restriction exact sequence

Let G be a group, N a normal subgroup and let A be a G-module. There is a long exact sequence

$$0 \to H^1(G/N, A^N) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(N, A)^{G/N} \xrightarrow{\text{trans}} H^2(G/N, A^N) \xrightarrow{\text{inf}} H^2(G, A),$$

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where A^N denotes the fixed points, and G/N acts on $H^1(N, A)$ by sending a 1-cocycle $f : N \to A$ to $(g * f)(n) = g \cdot f(g^{-1}ng)$.

Applying the inflation–restriction exact sequence, I

We apply this with $G = G_{\mathbb{Q}}$, N the normal subgroup of $G_{\mathbb{Q}}$ corresponding to the bicyclic extension $M := \mathbb{Q}(\chi, \chi')$ and $A = \mathbb{F}_\ell$ with trivial action.

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In this case we have

$$\begin{aligned} H^1(N, A)^{G/N} &= \mathrm{Hom}(G_M, \mathbb{F}_\ell)^{\mathrm{Gal}(M/\mathbb{Q})} \\ &= \{\rho : G_M \to \mathbb{F}_\ell : \rho(\sigma\tau\sigma^{-1}) = \rho(\tau) \text{ for } \tau \in G_M, \sigma \in G_{\mathbb{Q}}\}. \end{aligned}$$

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Using ρ to identify $\mathrm{Gal}(M(\rho)/M) \cong \mathbb{F}_\ell$, we naturally get a class in $H^2(\mathrm{Gal}(M/\mathbb{Q}), \mathbb{F}_\ell)$. This is the map trans.

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Recall that

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We do not want all elements $\rho \in \mathrm{Hom}(G_M, \mathbb{F}_\ell)^{\mathrm{Gal}(M/\mathbb{Q})}$, but only those corresponding to Heisenberg extensions.

These are precisely those ρ that map to a non-trivial multiple of the 2-cocycle $\theta_{\chi,\chi'}$ given by $(v, w) \mapsto \chi(v) \cdot \chi'(w)$.

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The first terms of the exact sequence are not too interesting: if we twist ρ by $\widetilde{\chi} : G_\mathbb{Q} \to \mathbb{F}_\ell$ (i.e. consider $\rho + \widetilde{\chi}$), we get another invariant character that maps to the same element in $H^2(\mathrm{Gal}(M/\mathbb{Q}), \mathbb{F}_\ell)$.

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Furthermore, the characters χ and χ' are trivial when restricted to M.

Applying the inflation–restriction exact sequence, III

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By exactness, these are precisely those $\theta_{\chi,\chi'}$ that vanish in $H^2(G_{\mathbb{Q}}, \mathbb{F}_\ell)$ or equivalently in $H^2(G_{\mathbb{Q}_v}, \mathbb{F}_\ell)$ for all places v of $\mathbb{Q}$ by class field theory.

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Theorem 3 (Realizing Heis_ℓ as Galois group)

There exists a Heisenberg extension containing M if and only if all ramified primes (not equal to ℓ) of M have residue field degree 1.

Questions?

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Happy April Fools' Day!