Counting nilpotent extensions

Peter Koymans
University of Michigan

Number Theory Web Seminar

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Malle's conjecture

Conjecture (Malle's conjecture)

Let G be a finite, non-trivial group. Then there exist numbers $a(G) \in \mathbb{Q}_{>0}$, $b(G) \in \mathbb{Z}_{\geq 0}$ and $c(G) > 0$ such that

$$\#\{K/\mathbb{Q} : D_K \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\} \sim c(G) X^{a(G)} (\log X)^{b(G)}.$$

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Sometimes $c(G)$ is an Euler product. This is expected to be true for S_n (Malle–Bhargava principle).

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- direct products $S_n \times A$ for $n \in \{3, 4, 5\}$ and A abelian by Wang (with $\#A$ coprime to some values) and later by Masri–Thorne–Tsai–Wang.

An exercise about hyperbolas

We have

$$\sum_{ab^2 \le X} 1 = \sum_{b \le \sqrt{X}} \sum_{a \le X/b^2} 1 = \sum_{b \le \sqrt{X}} \left(\frac{X}{b^2} + O(1) \right)$$
$$= X \sum_{b=1}^{\infty} \frac{1}{b^2} + O(\sqrt{X}).$$

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Compare instead with

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Both observations fail now.

Ramification theory

Let $K/\mathbb{Q}$ be a Galois extension and suppose that p does not divide $[K:\mathbb{Q}]$. Then

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Moral: inertia subgroups tend to "typically" be as small as possible when counting by discriminant.

An example

Example (Non-Galois quartic D_4)

If $L/\mathbb{Q}$ is quartic D_4 with quadratic subfield K, then for all $p \neq 2$

$$v_p(D_L) = \begin{cases} 3 & \text{if } p \text{ is totally ramified} \\ 2 & \text{if } p \text{ is in all other cases} \\ 1 & \text{if } p \text{ is unramified in } K/\mathbb{Q} \text{ but ramifies in the biquadratic} \\ 0 & \text{if } p \text{ is unramified.} \end{cases}$$

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Observations:

- main contribution comes from quadratic fields K with $D_K < \log\log\log\log X$;
- a positive proportion of the quartic D_4-extensions have a given quadratic field K as their subfield.

Difficulties with discriminant counting

Group theoretic properties greatly influence how difficult it is to count by discriminant, heavily exploited in previous works.

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Example ($L/\mathbb{Q}$ Galois with $\mathrm{Gal}(L/\mathbb{Q}) \cong D_{2^n}$)

Note that $D_{2^n} = \mathbb{Z}/2^n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The elements of minimal order are $(k, 1)$ (reflections) and $(2^{n-1}k, 0)$ (rotations with order dividing 2).

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\begin{tikzcd}
 & L & \\
 & L^{\langle (2^{n-1},0)\rangle} \arrow[u, dash, dashed] & \\
 & \mathbb{Q}(\sqrt{a},\sqrt{b}) = L^{\langle (2,0)\rangle} \arrow[u, dash, dashed] & \\
\mathbb{Q}(\sqrt{a}) \arrow[ur, dash] & \mathbb{Q}(\sqrt{ab}) = L^{\langle (1,0)\rangle} \arrow[u, dash, dashed] & \mathbb{Q}(\sqrt{b}) \arrow[ul, dash] \\
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& \mathbb{Q} &
\end{tikzcd}$$

Positive proportion of extensions have $L^{\langle(2^{n-1},0)\rangle}/\mathbb{Q}(\sqrt{ab})$ *unramified. So at least as hard as getting distribution of* $\mathrm{Cl}(\mathbb{Q}(\sqrt{d}))[2^\infty]$.

Fair counting functions

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Altug–Shankar–Varma–Wilson (2017): Malle’s conjecture for D_4 by Artin conductor.

Main result

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Theorem (K.–Pagano)

Assume GRH. Let G be a nilpotent group with $\#G$ odd. Then

$$\liminf_{X\to\infty} \frac{\#\left\{K/\mathbb{Q} : \prod_{p:I_p\neq\{\mathrm{id}\}} p \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\right\}}{c'(G)X(\log X)^{b'(G)}} \geq 1,$$

where $c'(G)$ is the expected Euler product and where $b'(G)$ is the naïve analogue of Malle's $b(G)$ in this situation.

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Surprisingly, the corresponding asymptotic

$$\lim_{X \to \infty} \frac{\#\left\{K/\mathbb{Q} : \prod_{p : I_p \neq \{\mathrm{id}\}} p \leq X, \mathrm{Gal}(K/\mathbb{Q}) \cong G\right\}}{c'(G) X (\log X)^{b'(G)}} = 1$$

is not true in general. Counterexamples exist for nilpotency class 2.

Applications

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Let $G_2, G_3 \twoheadrightarrow G_1$ be p-groups with p odd. TFAE:

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(i) *For every diagram*

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$$(\exists \overline{g_2}, \overline{h_2} \in G_2 : \overline{h_2}\,\overline{g_2}\,\overline{h_2}^{-1} = \overline{g_2}^{\alpha}) \Rightarrow (\exists \overline{g_3}, \overline{h_3} \in G_3 : \overline{h_3}\,\overline{g_3}\,\overline{h_3}^{-1} = \overline{g_3}^{\alpha}),$$

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This is known as the Massey vanishing conjecture (recently proven by Harpaz–Wittenberg for all p and all number fields).

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For 2-groups, the situation is much more involved. The only known proof is a famous result of Shafarevich (inverse Galois for solvable groups).

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Suppose that we have a H-extension $\pi : G_{\mathbb{Q}} \to H$, and consider

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{F}_p \arrow[r] & G \arrow[r] & H \arrow[r] & 1 \\ & & G_{\mathbb{Q}} \arrow[u, dashed, "?"'] \arrow[ur, "\pi"'] & & \end{tikzcd}$$

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$$\begin{tikzcd} 1 \arrow[r] & \mathbb{F}_p \arrow[r] & G \arrow[r] & H \arrow[r] & 1 \\ & & G_{\mathbb{Q}} \arrow[u, dashed, "?"'] \arrow[ur, "\pi"'] & & \end{tikzcd}$$

It is well-known that we have a local-to-global for the above diagram, which means that we have to control $\pi(\mathrm{Frob}_v)$ for all v.

Scholz–Reichardt sketch II

Note that H also fits in an exact sequence

$$1 \to \mathbb{F}_p \to H \to H' \to 1.$$

Therefore we may twist our H-extension $\pi : G_{\mathbb{Q}} \to H$ by $\chi : G_{\mathbb{Q}} \to \mathbb{F}_p$ to get $\pi + \chi : G_{\mathbb{Q}} \to H$.

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Here we use that p is odd in an essential way: $\chi_\ell(\mathrm{Frob}_q)$ and $\chi_q(\mathrm{Frob}_\ell)$ are independent.

Step 1a: parametrization

To give an idea how the techniques work, we will (unconditionally!) give an overview for the proof of the asymptotic for the number of Galois D_4-extensions by product of ramified primes.

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We have a central exact sequence

$$0 \to \mathbb{F}_2 \to D_4 \xrightarrow{q} \mathbb{F}_2^2 \to 0$$

and a bijection

$$\mathrm{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2^2) \leftrightarrow \{(a, b) \in (\mathbb{Q}^*/\mathbb{Q}^{*2})^2 : a, b \text{ lin. ind.}\}.$$

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It is well-known that a $\mathbb{F}_2^2$-extension $\mathbb{Q}(\sqrt{a}, \sqrt{b})$ of $\mathbb{Q}$ is contained in a D_4-extension if and only if $x^2 = ay^2 + bz^2$ has a non-trivial point.

Step 1b: parametrization

If $\rho \in \mathrm{Epi}(G_{\mathbb{Q}}, D_4)$ is a lift of $\pi \in \mathrm{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2)$ and $q : D_4 \twoheadrightarrow \mathbb{F}_2^2$, then

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The variables α_S are squarefree and pairwise coprime, and we have $\mathrm{rad}(|abc|) = \prod_{\emptyset \subset S \subseteq \{a,b,c\}} |\alpha_S|$.

Step 2: character sums

Define $T(a)$ to be the subsets of $\{a, b, c\}$ containing a. Then we have

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Hasse-Minkowski: detect solubility of conic locally at primes dividing α_S.

Now rewrite the above sum as a sum over Legendre symbols involving the variables α_S.

Step 3: equidistribution

Evaluate the resulting character sum using Chebotarev and the large sieve.

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- Proof can most likely be made unconditional with a suitably strong large sieve for nilpotent extensions.

Questions?

Thank you for your attention!