# The negative Pell equation and Stevenhagen's conjecture


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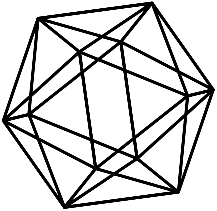

*Fachbeirat*

7 July 2021

# History of Pell's equation

For a fixed squarefree integer $d > 0$, the equation

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Lagrange was the first to give an algorithm with proof of correctness.

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Question: as we vary $d$, how often is the negative Pell equation soluble?

# Solubility over the rationals

Define $\mathcal{D}$ to be the set of squarefree integers having as odd prime divisors only primes $p \equiv 1 \bmod 4$ and define $\mathcal{D}^-$ to be the set of squarefree integers for which negative Pell is soluble.

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Question: what is the density of $\mathcal{D}^-$ inside $\mathcal{D}$?

# Conjectures on the negative Pell equation

Nagell (1930s) conjectured that

$$\lim_{X \to \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}}$$

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Stevenhagen (1995) conjectured that

$$\lim_{X\to\infty} \frac{\#\{d \le X : d \in \mathcal{D}^-\}}{\#\{d \le X : d \in \mathcal{D}\}} = 1 - \alpha,$$

where

$$\alpha = \prod_{j=1}^{\infty} (1 + 2^{-j})^{-1} \approx 0.41942.$$

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$$\frac{5\alpha}{4} \leq \liminf_{X\to\infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} \leq \limsup_{X\to\infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} \leq \frac{2}{3}.$$

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Together with Stephanie Chan, Djordjo Milovic and Carlo Pagano, I improved the lower bound to

$$\alpha \cdot \sum_{n=0}^{\infty} 2^{-n(n+3)/2} \approx \alpha \cdot 1.28325.$$

# Stevenhagen's conjecture is true

**Theorem 1 (K., Pagano (2021))**

*We have*

$$\lim_{X \to \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} = 1 - \alpha$$

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**Corollary 2**

*We have*

$$\#\{d \leq X : d \in \mathcal{D}^-\} \sim C \cdot (1 - \alpha) \cdot \frac{X}{\sqrt{\log X}}.$$

# A criterion for solubility

Recall that the narrow class group $\mathrm{Cl}^+(K)$ is defined as the quotient of the ideal group $I_K$ by the principal ideals $P_K^+$ admitting a totally positive generator, while the class group is the quotient by the principal ideals $P_K$.

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We have

$$\begin{aligned} x^2 - dy^2 = -1 \text{ is soluble } &\Leftrightarrow \text{ fundamental unit } \epsilon \text{ has negative norm} \\ &\Leftrightarrow (\sqrt{d}) \text{ is trivial in } \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})). \end{aligned}$$

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There is a fundamental exact sequence

$$1 \to \frac{P_K}{P_K^+} \to \mathrm{Cl}^+(K) \to \mathrm{Cl}(K) \to 1$$

with $\#\frac{P_K}{P_K^+} \in \{1, 2\}$ and $\frac{P_K}{P_K^+}$ generated by $(\sqrt{d})$.

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Goal: study joint distribution of $(\mathrm{Cl}^+(K)[2^\infty], \mathrm{Cl}(K)[2^\infty])$.

# Genus theory

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The description of $\mathrm{Cl}^+(K)[2]$ is due to Gauss and is known as genus theory. We have that

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and $\mathrm{Cl}^+(K)[2]$ is generated by the ramified prime ideals of $\mathcal{O}_K$.

If $p$ divides the discriminant of $\mathbb{Q}(\sqrt{d})$, then $p$ ramifies, so

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{d}) & \mathfrak{p} & \mathfrak{p}^2 = (p). \\ \Big| & \Big| & \\ \mathbb{Q} & p & \end{array}$$

There is precisely one relation between the ramified primes.

# Cohen–Lenstra–Gerth

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More formally, Cohen and Lenstra conjectured that

$$\lim_{X\to\infty} \frac{\#\left\{K \text{ im. quadr.} : |D_K| < X \text{ and } \mathrm{Cl}(K)[p^\infty] \cong A\right\}}{\#\left\{K \text{ im. quadr.} : |D_K| < X\right\}} = \frac{\prod_{i=1}^{\infty}\left(1 - \frac{1}{p^i}\right)}{\#\mathrm{Aut}(A)}$$

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**Theorem 3 (Alexander Smith (2017))**

*Gerth's conjecture is true.*

# Strategy for Stevenhagen's conjecture

**Example 1 (Definition of $2^k$-rank)**

*Take*

$$A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \quad .$$

*Then* $\mathrm{rk}_2 A = 3$, $\mathrm{rk}_4 A = \mathrm{rk}_8 A = 1$, $\mathrm{rk}_{2^k} A = 0$ *for* $k > 3$.

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Find for every integer $m \geq 1$, the density of $d \in \mathcal{D}$ for which

$$\mathrm{rk}_{2^k} \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = \mathrm{rk}_{2^k} \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) > 0 \text{ for } 1 \leq k \leq m \text{ and}$$

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Strategy: adapt Smith's ideas to compute these densities.

# Duality of abelian groups

For a finite abelian group $A$, define

$$A^\vee := \mathrm{Hom}(A, \mathbb{C}^*).$$

There is a natural pairing

$$\mathrm{Art}_1 : A[2] \times A^\vee[2] \to \{\pm 1\}, \quad (a, \chi) \mapsto \chi(a).$$

Left kernel of $\mathrm{Art}_1$ is $2A[4]$ and right kernel is $2A^\vee[4]$.

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Goal: in order to compute 4-rank, understand $\mathrm{Art}_1$. We start by describing $\mathrm{Cl}^{+,\vee}(K)[2]$.

# The dual class group

**Theorem 4 (Class field theory)**

*We have an isomorphism*

$$\mathrm{Cl}^+(K) \cong \mathrm{Gal}(H^+(K)/K)$$

*given by sending a prime ideal* $\mathfrak{p}$ *to* $\mathrm{Art}(\mathfrak{p})$*. Furthermore, if* $K$ *is Galois, this isomorphism respects the natural Galois action of* $\mathrm{Gal}(K/\mathbb{Q})$ *on both sides.*

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If $K$ is quadratic with odd discriminant, then $\mathrm{Cl}^{+,\vee}(K)[2]$ is generated by the quadratic characters $\chi_{p^*}$, where $p^*$ satisfies $|p^*| = |p|$ and $p^* \equiv 1 \bmod 4$. In particular

$$d \in \mathcal{D} \iff \mathrm{rk}_2 \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = \mathrm{rk}_2 \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \iff (\sqrt{d}) \in 2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d}))[4].$$

# The Artin pairing

Under the earlier identifications, we have that

$$\mathrm{Art}_1 : \mathrm{Cl}^{+}(K)[2] \times \mathrm{Cl}^{+,\vee}(K)[2] \to \{\pm 1\}, \quad (\mathfrak{p}, \chi) \mapsto \chi(\mathrm{Art}\ \mathfrak{p}).$$

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Let $p_1, \ldots, p_t$ be the prime divisors of $d$. The Rédei matrix is

$$\begin{array}{ccccc} & \chi_{p_1^*} & \chi_{p_2^*} & \cdots & \chi_{p_t^*} \\ p_1 & * & \left(\frac{p_2}{p_1}\right) & \cdots & \left(\frac{p_t}{p_1}\right) \\ p_2 & \left(\frac{p_1}{p_2}\right) & * & \cdots & \left(\frac{p_t}{p_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_t & \left(\frac{p_1}{p_t}\right) & \left(\frac{p_2}{p_t}\right) & \cdots & * \end{array}.$$

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Left kernel gives a generating set for $2\mathrm{Cl}^{+}(K)[4]$.

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For $d \in \mathcal{D}$, recall that $(\sqrt{d}) \in 2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d}))[4]$.

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Heuristic assumption: every non-zero element in the generating set of $2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d}))[4]$ is equally likely to be trivial.

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Heuristic assumption: every non-zero element in the generating set of $2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d}))[4]$ is equally likely to be trivial.

**Conjecture 1 (Stevenhagen's conjecture)**

*We have*

$$\lim_{X\to\infty} \frac{\#\{d \le X : d \in \mathcal{D}^-\}}{\#\{d \le X : d \in \mathcal{D}\}} = \sum_{j=0}^{\infty} \frac{\mathbb{P}(\text{4-rank of } d \in \mathcal{D} \text{ equals } j)}{2^{j+1}-1}.$$

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Furthermore,

$$\mathbb{P}(4\text{-rank of } d \in \mathcal{D} \text{ equals } j) = \lim_{t \to \infty} \mathbb{P}(t \times t \text{ sym. matrix has ker. of dim. } j).$$

# The second Artin pairing

There is a natural pairing

$$\mathrm{Art}_2 : 2A[4] \times 2A^\vee[4] \to \{\pm 1\}, \quad (a, \chi) \mapsto \psi(a),\ 2\psi = \chi.$$

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Fact: a degree 4 unramified, abelian extension of $\mathbb{Q}(\sqrt{d})$ is Galois over $\mathbb{Q}$ with Galois group $D_4$.

Such extensions are of the shape $\mathbb{Q}(\sqrt{d}, \sqrt{a}, \sqrt{\alpha})$, where

$$x^2 = ay^2 + \frac{d}{a}z^2 \text{ with } x, y, z \in \mathbb{Z} \text{ and } \gcd(x, y, z) = 1, \quad \alpha := x + y\sqrt{a}.$$

# Reflection principles

In the literature there are many known results that compare different class groups. For example, we have

$$\mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \leq \mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{-3d})) \leq 1 + \mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{d})),$$

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How can we find such reflection principles?

Smith's idea is to look for situations where the compositum of various Hilbert class fields is in some sense *small*.

# Intersections of Hilbert class fields

Take primes $p_1, p_2, q_1, q_2$. Now suppose that we have a degree 4 unramified, abelian extension of $\mathbb{Q}(\sqrt{dp_i q_j})$ each lifting the character $\chi_a$.

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Recall that we then get $\alpha_{i,j} \in \mathbb{Q}(\sqrt{a})$ with

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In other words, part of $H_2(\mathbb{Q}(\sqrt{dp_2q_2}))$ is contained in the other $H_2(\mathbb{Q}(\sqrt{dp_iq_j}))$. This implies

$$\text{Art}_{2,dp_1q_1}(b,\chi_a)+\text{Art}_{2,dp_1q_2}(b,\chi_a)+\text{Art}_{2,dp_2q_1}(b,\chi_a)+\text{Art}_{2,dp_2q_2}(b,\chi_a) = 0$$

for $b \in 2\text{Cl}(\mathbb{Q}(\sqrt{dp_iq_j}))[4]$ a fixed divisor of $d$.

# Another reflection principle

With similar techniques, Smith proves another reflection principle

$$\begin{aligned}&\mathrm{Art}_{2,dp_1q_1}(b,\chi_{ap_1})+\mathrm{Art}_{2,dp_1q_2}(b,\chi_{ap_1})+\\&\qquad\mathrm{Art}_{2,dp_2q_1}(b,\chi_{ap_2})+\mathrm{Art}_{2,dp_2q_2}(b,\chi_{ap_2})=\sum_{r\mid b}\mathrm{Frob}_{K_{p_1p_2,q_1q_2}/\mathbb{Q}}(r).\end{aligned}$$

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The above reflection principle is useless in both cases.

We develop two new reflection principles. Unlike Smith's work, they make essential use of Hilbert reciprocity in multiquadratic fields.

# Bonus slide: new reflection principles

For the Artin pairing with $dp_iq_j$ we have (following Smith's ideas)

$$\mathrm{Art}_{2,dp_1q_1}(dp_1q_1, \chi_{ap_1}) + \mathrm{Art}_{2,dp_1q_2}(dp_1q_2, \chi_{ap_1}) + \mathrm{Art}_{2,dp_2q_1}(dp_2q_1, \chi_{ap_2}) + \mathrm{Art}_{2,dp_2q_2}(dp_2q_2, \chi_{ap_2}) = \mathrm{Frob}_{K_{p_1p_2,q_1q_2}/\mathbb{Q}}(\infty).$$

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Our reciprocity law shows that

$$\mathrm{Frob}_{K_{p_1p_2,q_1q_2}/\mathbb{Q}}(\infty)=\mathrm{Frob}_{K_{p_1p_2,-1}/\mathbb{Q}}(q_1)+\mathrm{Frob}_{K_{p_1p_2,-1}/\mathbb{Q}}(q_2).$$

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For the pairing between $a$ and $\chi_a$ we also develop a new reflection principle.

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Thank you for your attention!