

Spins, Galois representations and a question of Ramakrishna

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A question about $a_p(E)$

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If $p \equiv 3 \pmod{4}$ and $p > 3$, then E has supersingular reduction at p , so

$$a_p(E) = 0,$$

which is a cube modulo p .

Rephrasing the question

Theorem (Cox, Theorem 14.16)

Let K be an imaginary quadratic field and let \mathcal{O} be an order in K . Let L be the ring class field of \mathcal{O} , and let E be an elliptic curve over L with $\text{End}_{\mathbb{C}}(E) = \mathcal{O}$.

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Let p be a prime that splits completely in L and \mathfrak{p} be a prime in K above p . Suppose E has good reduction at \mathfrak{p} . Then there is $\kappa \in \mathcal{O}$ such that $p = \kappa \bar{\kappa}$ and

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$$E : y^2 = x^3 - x, \quad K = \mathbb{Q}(i), \quad \mathcal{O} = \mathbb{Z}[i], \quad L = \mathbb{Q}(i).$$

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Since $\mathbb{Z}[i]$ is a PID, we can write $p = \pi\bar{\pi}$, and π is unique up to multiplying by a power of i . Then, for some choice of π , we have

$$a_p(E) = \pi + \bar{\pi}.$$

A criterion for $a_p(E)$ being a cube modulo p

Since we have $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}[i]/\pi\mathbb{Z}[i]$, we have that $a_p(E) = \pi + \bar{\pi}$ is a cube modulo p if and only if it is a cube modulo π .

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Lemma

If $p \equiv 1 \pmod{12}$, then $\bar{\pi}$ is a cube modulo π if and only if $\overline{i \cdot \pi}$ is a cube modulo π .

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Since $p \equiv 1 \pmod{12}$, we know that there is a primitive 12-th root of unity ζ_{12} in \mathbb{F}_p . Thus $i = \zeta_4$ is a cube in \mathbb{F}_p . \square

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Corollary

Let $p \equiv 1 \pmod{12}$. Then $a_p(E)$ is a cube modulo p if and only if $(\bar{\pi}/\pi)_3 = 1$, where π is any element of $\mathbb{Z}[i]$ satisfying $\pi\bar{\pi} = p$.

Cubic residue symbols

Let K be a number field with $\zeta_3 \in K$. For $\alpha \in \mathcal{O}_K$ and $\mathfrak{p} \nmid 3\mathcal{O}_K$ a prime, we define $(\alpha/\mathfrak{p})_{K,3}$ as the unique element in $\{1, \zeta_3, \zeta_3^2, 0\}$ with

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,3} \equiv \alpha^{\frac{N_{K/\mathbb{Q}}(\mathfrak{p})-1}{3}} \pmod{\mathfrak{p}}.$$

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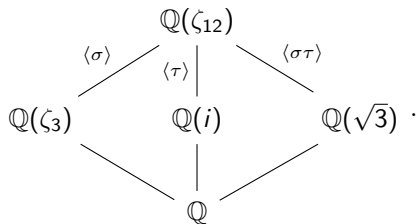
$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{K,3} \equiv \alpha^{\frac{N_{K/\mathbb{Q}}(\mathfrak{p})-1}{3}} \pmod{\mathfrak{p}}.$$

We multiplicatively extend this to all ideals coprime to 3.

This residue symbol has the same usual properties as the quadratic residue symbol, i.e. periodicity and reciprocity.

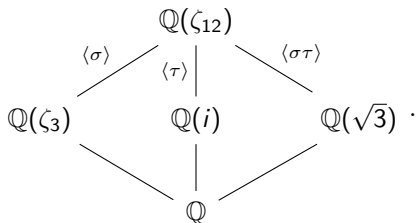
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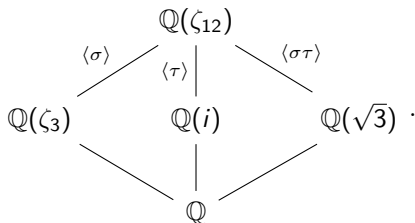
For an ideal \mathfrak{a} of $\mathbb{Z}[\zeta_{12}]$, we define the symbol $[\mathfrak{a}]$

$$[\mathfrak{a}] := \begin{cases} \left(\frac{\sigma(\alpha)\sigma\tau(\alpha)}{\alpha} \right)_{\mathbb{Q}(\zeta_{12}),3} & \text{if } \gcd(\mathfrak{a}, (3)) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

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where α is any generator of \mathfrak{a} . The symbol is well-defined, and satisfies

$$\sum_{\rho \in \text{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})} [\rho(\mathfrak{p})] = \begin{cases} -2 & \text{if } a_p(E) \text{ is not a cube modulo } p \\ 4 & \text{if } a_p(E) \text{ is a cube modulo } p \end{cases}$$

for \mathfrak{p} a split prime of degree 1 (i.e. $p = \mathfrak{p} \cap \mathbb{Z}$ satisfies $p \equiv 1 \pmod{12}$).

Our main results

Theorem (K.-Uttenthal)

There exists $C > 0$ such that for all $X \geq 100$

$$\left| \sum_{\substack{N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\mathfrak{p}) \leq X \\ \mathfrak{p} \text{ prime}}} [\mathfrak{p}] \right| \leq CX^{\frac{3199}{3200}}.$$

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Corollary (K.-Uttenthal)

We have

$$\frac{\#\{p \equiv 1 \pmod{12} : a_p(E) \text{ is a cube modulo } p\}}{\#\{p \equiv 1 \pmod{12}\}} = \frac{1}{3} + O\left(\frac{\log X}{X^{1/3200}}\right).$$

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In fact, one can prove a similar result for any imaginary quadratic field, which has applications to a conjecture of Weston.

Spin of prime ideals

Definition

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Given $\sigma \in \text{Gal}(K/\mathbb{Q})$ and a principal prime \mathfrak{p} of K admitting a totally positive generator, FIMR define

$$\text{spin}(\sigma, \mathfrak{p}) = \left(\frac{\sigma(\pi)}{\mathfrak{p}} \right)_{K,2},$$

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where $(\cdot/\cdot)_{K,2}$ is the quadratic residue symbol in K and where π is any totally positive generator of \mathfrak{p} . This is well-defined, as changing the generator π of \mathfrak{p} changes π by the square of a unit.

The main result of FIMR

Theorem (FIMR)

Assume that K/\mathbb{Q} is cyclic of degree n and that σ is a generator of $\text{Gal}(K/\mathbb{Q})$. If $n \geq 4$, assume a short character sum conjecture. There exists $\delta > 0$ such that for all $X \geq 100$

$$\left| \sum_{N_{K/\mathbb{Q}}(\mathfrak{p}) \leq X} \text{spin}(\sigma, \mathfrak{p}) \right| \ll X^{1-\delta}.$$

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We adapt their arguments to cubic residue symbols and the field $K = \mathbb{Q}(\zeta_{12})$, which is neither cyclic nor totally real, and has degree ≥ 4 .

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We adapt their arguments to cubic residue symbols and the field $K = \mathbb{Q}(\zeta_{12})$, which is neither cyclic nor totally real, and has degree ≥ 4 .

Our main analytic achievement is in making their techniques unconditional in this case.

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- ▶ lifting problems of Galois representations.

Proving oscillation of spins: Vinogradov's sieve

Vinogradov's sieve is the only sieve at the moment that is able to catch primes. We will discuss it for simplicity over \mathbb{Z} .

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Theorem (Vinogradov's sieve)

Let y_n be a sequence indexed by positive integers such that $y_p = a_p$ for all primes p . Assume that we have good estimates for

$$\sum_{\substack{n \leq X \\ n \equiv 0 \pmod{q}}} y_n \quad (\text{sums of type I, linear sums})$$

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$$\sum_{n \leq X} \sum_{m \leq Y} \alpha_n \beta_m y_{nm} \quad (\text{sums of type II, bilinear sums})$$

for all $\alpha_n, \beta_m \in \mathbb{C}$ bounded by 1.

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Then we get an estimate for $\sum_{p \leq X} a_p$.

Extending the sequence

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There are natural candidates for this both in our problem, namely the symbol $[\mathfrak{a}]$, and also in FIMR, namely

$$\text{spin}(\sigma, \mathfrak{a}) = \left(\frac{\sigma(\alpha)}{\mathfrak{a}} \right)_{K,2},$$

where α is a totally positive generator of \mathfrak{a} .

Sums of type II

Let $\alpha_n, \beta_m \in \mathbb{C}$ be bounded by 1. The bilinear sums

$$\sum_{N_{K/\mathbb{Q}}(\mathfrak{n}) \leq X} \sum_{N_{K/\mathbb{Q}}(\mathfrak{m}) \leq Y} \alpha_n \beta_m \text{spin}(\sigma, \mathfrak{nm})$$

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Indeed, the key point is the “twisted multiplicativity” of spin

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The twist factor $t(\mathfrak{n}, \mathfrak{m})$ can be computed explicitly and roughly looks like the Legendre symbol

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Absorbing $\text{spin}(\sigma, \mathfrak{n})$ and $\text{spin}(\sigma, \mathfrak{m})$ in the coefficients α_n and β_m , it suffices to estimate

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This can be handled using large sieve techniques.

Sums of type I

The essential difficulty lies in the estimation of sums of type I. These are

$$\sum_{\substack{N_{K/\mathbb{Q}}(\mathfrak{a}) \leq X \\ \mathfrak{a}=(\alpha), \alpha \text{ tot. pos.}}} \left(\frac{\sigma(\alpha)}{\alpha} \right)_{K,2},$$

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The insight of FIMR is to approach this as follows: we split

$$O_K = \mathbb{Z} \oplus \mathbb{M},$$

so $\alpha = a + \beta$ with $a \in \mathbb{Z}$, $\beta \in \mathbb{M}$.

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where we have taken $q = 1$ for simplicity.

The insight of FIMR is to approach this as follows: we split

$$O_K = \mathbb{Z} \oplus \mathbb{M},$$

so $\alpha = a + \beta$ with $a \in \mathbb{Z}$, $\beta \in \mathbb{M}$.

Then we have $\sigma(\alpha) = a + \sigma(\beta)$, hence

$$\left(\frac{\sigma(\alpha)}{\alpha} \right)_{K,2} = \left(\frac{a + \sigma(\beta)}{a + \beta} \right)_{K,2} = \left(\frac{\sigma(\beta) - \beta}{a + \beta} \right)_{K,2} \approx \left(\frac{a + \beta}{\sigma(\beta) - \beta} \right)_{K,2}.$$

Sums of type I, continued

Recall that $O_K = \mathbb{Z} \oplus \mathbb{M}$, $\alpha = a + \beta$ and

$$\left(\frac{\sigma(\alpha)}{\alpha} \right)_{\kappa,2} = \left(\frac{a + \beta}{\sigma(\beta) - \beta} \right)_{\kappa,2}.$$

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We now fix β , then a runs over a sum of typical length $X^{1/n}$, while the conductor is $N_{K/\mathbb{Q}}(\sigma(\beta) - \beta)$ typically of size X . So our sum is “short”. Here is where the short character sum conjecture comes in.

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Technical warning: to make this precise, note that every ideal \mathfrak{a} has infinitely many generators. So to avoid our sums running over infinitely many terms, we need to construct a fundamental domain and pick for each ideal \mathfrak{a} the unique generator from the fundamental domain.

Why does FIMR only allow cyclic Galois groups?

The character $\left(\frac{a+\beta}{\sigma(\beta)-\beta}\right)_{\kappa,2}$ does not oscillate if $\sigma(\beta) - \beta$ is a square.

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This difficulty was overcome by K.-Milovic, who also obtained the joint distribution of spins

$$\prod_{\sigma \in S} \text{spin}(\sigma, \mathfrak{p}),$$

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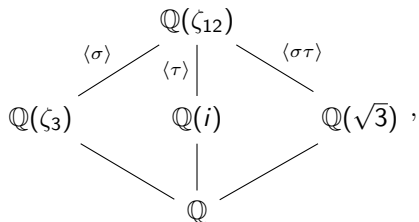
for any subset S of $\text{Gal}(K/\mathbb{Q})$ satisfying $\sigma \in S \Rightarrow \sigma^{-1} \notin S$. This assumption is important because

$$\text{spin}(\sigma, \mathfrak{p}) = \left(\frac{\sigma(\pi)}{\pi}\right), \quad \text{spin}(\sigma^{-1}, \mathfrak{p}) = \left(\frac{\sigma^{-1}(\pi)}{\pi}\right)$$

are related by quadratic reciprocity. This was further studied by McMeekin, and Chan–McMeekin–Milovic.

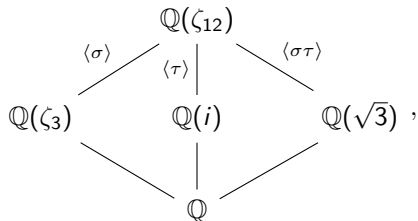
Back to our situation

Recall the field diagram



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so our aim is to estimate the type I sums

$$\sum_{\substack{\alpha \in \mathbb{Z}[\zeta_{12}] \\ N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\alpha) \leq X}} \left(\frac{\sigma(\alpha)\sigma\tau(\alpha)}{\alpha} \right)_{\mathbb{Q}(\zeta_{12}), 3}.$$

Field lowering

It turns out that the symbol

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Let us now show how this “field lowering” mechanism happens.

Field lowering in practice

We get from the FIMR method, writing $\alpha = a + \beta$

$$\left(\frac{\sigma\tau(\alpha)}{\alpha} \right)_{\mathbb{Q}(\zeta_{12}, 3)} \approx \left(\frac{a + \beta}{\sigma\tau(\beta) - \beta} \right)_{\mathbb{Q}(\zeta_{12}, 3)} .$$

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Furthermore, $a + \beta$ is fixed by $\sigma\tau$ modulo $\sigma\tau(\beta) - \beta$. Thus there is some $\gamma \in \mathbb{Z}[\sqrt{3}]$ such that

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We rewrite

$$\left(\frac{a + \beta}{\sigma\tau(\beta) - \beta} \right)_{\mathbb{Q}(\zeta_{12}, 3)} = \left(\frac{a + \beta}{\mathfrak{c}\mathbb{Z}[\zeta_{12}]} \right)_{\mathbb{Q}(\zeta_{12}, 3)} = \left(\frac{\gamma}{\mathfrak{c}\mathbb{Z}[\zeta_{12}]} \right)_{\mathbb{Q}(\zeta_{12}, 3)}.$$

The field lowering lemmas

Lemma (Field lowering for split primes)

Let K be a number field and let \mathfrak{p} be a prime of K coprime to 3. Assume that L is a quadratic extension of K such that L contains ζ_3 and \mathfrak{p} splits in L . Write σ for the non-trivial element of $\text{Gal}(L/K)$. Then for $\alpha \in \mathcal{O}_K$

$$\left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_L}\right)_{L,3} = \begin{cases} \left(\frac{\alpha}{\mathfrak{p}\mathcal{O}_K}\right)_{K,3}^2 & \text{if } \sigma \text{ fixes } \zeta_3 \\ \mathbf{1}_{\mathfrak{p} \nmid \alpha} & \text{if } \sigma \text{ does not fix } \zeta_3. \end{cases}$$

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Lemma (Field lowering for inert primes)

Let K be a number field and let \mathfrak{p} be a prime of K coprime to 3. Assume that L is a quadratic extension of K such that L contains ζ_3 and assume that \mathfrak{p} stays inert in L . Further assume that \mathfrak{p} has degree 1 in K and let p be the prime of \mathbb{Q} lying below \mathfrak{p} . Then we have for all $\alpha \in O_K$

$$\left(\frac{\alpha}{\mathfrak{p}O_L}\right)_{L,3} = \left(\frac{\alpha}{\mathfrak{p}O_K}\right)_{K,3}^{p+1}.$$

Thank you for your attention!