Statistics of Diophantine equations

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Overview

Diophantine equations and factoring

Class groups

Pell's equation

Future work

Fermat's last theorem

Pierre de Fermat

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$$X^n + Y^n = Z^n.$$

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We have proven Fermat's last theorem!

Unfortunately, not quite ...

Ernst Kummer

Consider the ring

$$\mathbb{Z}[\zeta_n] = \{a_0 + a_1\zeta_n + a_2\zeta_n^2 + \cdots + a_{n-1}\zeta_n^{n-1}\}.$$

We can formally add such expressions, and also multiply them using $\zeta_n^n = 1$.

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The class number measures unique factorization.

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Let R be a commutative domain. An element $\pi \in R$ is called irreducible if all divisors d of π satisfy $d = u$ or $d = \pi u$ for some unit u.

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The root cause of this problem is that the ideals

$$I = (2, \sqrt{-6}) \supsetneq 2\mathbb{Z}[\sqrt{-6}], \quad J = (3, \sqrt{-6}) \supsetneq 2\mathbb{Z}[\sqrt{-6}]$$

of $\mathbb{Z}[\sqrt{-6}]$ are not principal. If it were, we could use it to further factor 2, 3 and $\sqrt{-6}$.

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This definition also plays a key role in other areas of mathematics (Picard group, Jacobian etc.).

Why is the class group so important?

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Although Wiles' proof does not rely on factoring in any way, class field theory is essential to his whole approach!

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If one numerically enumerates d such that 9 exactly divides $|\mathrm{Cl}(\mathbb{Z}[\sqrt{-d}])|$, then one sees that the group $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is 8 times less likely than $\mathbb{Z}/9\mathbb{Z}$. Why?

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Conjecture (Cohen–Lenstra, 1984)

Let p be an odd prime. Let A be a finite abelian group such that all elements have order p^k for some k. Then

$$\lim_{X\to\infty} \frac{|\{0 < d < X \text{ sqf.} : \mathrm{Cl}(\mathbb{Z}[\sqrt{-d}])[p^\infty] \cong A\}|}{|\{0 < d < X : \text{ sqf.}\}|} = \frac{\prod_{i=1}^{\infty}\left(1 - \frac{1}{p^i}\right)}{|\mathrm{Aut}(A)|}$$

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Consider n labelled vertices. Among the set of all possible graphs with n labelled vertices, how many are isomorphic to a given graph $\mathcal{G}$? This is

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One can do a similar story for "random abelian groups" (i.e. random multiplication tables), and this produces $c/\mathrm{Aut}(A)$ for c a constant.

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Solution x, y	Ratio x/y	Expansion of $\sqrt{2}$
$x = 1,\ y = 1$	1	1.4142135...
$x = 3,\ y = 2$	1.5	1.4142135...
$x = 7,\ y = 5$	1.4	1.4142135...
$x = 17,\ y = 12$	1.4166666...	1.4142135...
$x = 41,\ y = 29$	1.4137931...	1.4142135...
$x = 99,\ y = 70$	1.4142857...	1.4142135...

The riddle of Archimedes

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The smallest solution after $x^2 = 1$, $y^2 = 0$ takes 50 pages to write down ($\approx 7.76 \times 10^{206544}$).

The positive Pell equation

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The smallest non-trivial solution to $x^2 - 61y^2 = 1$ *is*

$$x = 1766319049, \quad y = 226153980.$$

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Nagell (1930s) conjectured

$$\lim_{X \to \infty} \frac{\#\{d \leq X : d \in \mathcal{D}, x^2 - dy^2 = -1 \text{ sol.}\}}{\#\{d \leq X : d \in \mathcal{D}\}}$$

exists and lies in $(0, 1)$. The smallest $d \in \mathcal{D}$ for which the negative Pell equation is not soluble is $d = 34$.

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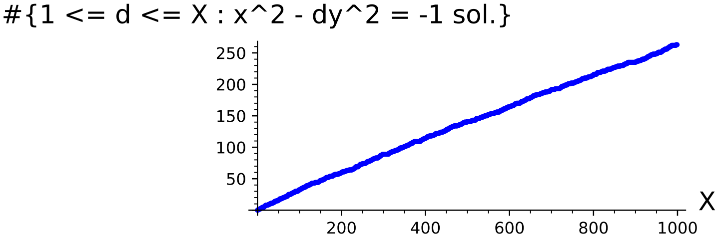
Stevenhagen (1995) predicted a precise value for the limit.

Proof of Stevenhagen's conjecture



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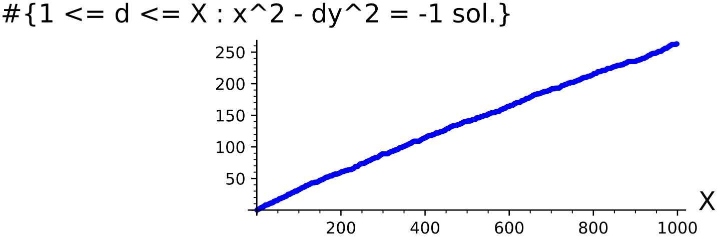

Frequency of negative Pell solubility

Theorem (K.–Pagano, 2022)

Stevenhagen's conjecture is true.

Translating to class groups

Consider the ring $\mathbb{Z}[\sqrt{d}]$. There is an automorphism $\sigma : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}[\sqrt{d}]$ given by $x + y\sqrt{d} \mapsto x - y\sqrt{d}$. Let $N(\alpha) = \alpha\sigma(\alpha)$. Note that

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Negative Pell equation can thus be solved if and only if there is a unit of norm -1.

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Rephrase this as an equality between the narrow class group (ideals modulo principal ideals with a totally positive generator) and the ordinary class group (ideals modulo principal ideals).

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We only need to consider the 2-Sylow since $(\sqrt{d})$ has order 2 in the narrow class group. This is the only part of the class group that is well-understood by a recent breakthrough of A. Smith.

Overview

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Example

For $n = 6$ one can use the factorization $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ to show that there are no integer solutions. However, we have

$$6 = \left(\frac{17}{21}\right)^3 + \left(\frac{37}{21}\right)^3.$$

Results

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Key tool: obtain distribution of Selmer group of $x^3 + y^3 = n$. This is the analogue of the class group for elliptic curves.

Chowla's conjecture

Sarvadaman Chowla

Kannan Soundararajan

Conjecture (Generalized Riemann hypothesis)

All non-trivial zeroes of $L(s, \chi)$ *lie on* $s = 1/2 + it$.

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There has also been great interest in the function field case of this conjecture.

Function fields

Theorem (W. Li (2018))

Let q be an odd prime power. There are infinitely many monic, squarefree polynomials $D \in \mathbb{F}_q[t]$ such that $L(\frac{1}{2}, \chi_D) = 0$.

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Theorem (K.–Pagano–Shusterman)

We have $L(\frac{1}{2}, \chi_D) \neq 0$ for 100% of the monic squarefree polynomials D.

Proof sketch

By a result of Grothendieck we have $L(\frac{1}{2}, \chi_D) \neq 0$ if and only if there exists an embedding

$$\mathbb{Q}_2/\mathbb{Z}_2 \hookrightarrow \mathrm{Jac}(C_D)(\overline{\mathbb{F}_q})[2^\infty][\mathrm{Frob}_q^2 - q],$$

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The Jacobian can be viewed as a function field analogue of the class group.

A suitable adaptation of our methods for Stevenhagen's conjecture allow one to obtain the distribution of this Jacobian, from which the theorem follows.

Questions?

Thank you for your attention!