Value sets of binary forms

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The value set

Definition (Value set)

Let $F \in \mathbb{Z}[X, Y]$ be a binary form (i.e. homogeneous polynomial in two variables). Define

$$\mathrm{Val}(F) := \{F(x, y) : (x, y) \in \mathbb{Z}^2\}.$$

For two forms $F, G \in \mathbb{Z}[X, Y]$, we say $F \sim_{\mathsf{val}} G$ if $\mathrm{Val}(F) = \mathrm{Val}(G)$.

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Example (Fermat)

We have

$$\mathrm{Val}(X^2 + Y^2) = \{n \in \mathbb{Z}_{>0} : p \mid n \text{ and } p \equiv 3 \bmod 4 \Rightarrow v_p(n) \equiv 0 \bmod 2\}.$$

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Class field theory gives an explicit description of $\mathrm{Val}(F)$ for F binary quadratic. However, much less is known if $\deg(F) \geq 3$.

Equivalence of forms

Recall that two binary forms $F, G \in \mathbb{Z}[X, Y]$ are $GL_2(\mathbb{Z})$–equivalent, written $F \sim_{GL_2(\mathbb{Z})} G$, if there exists $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ with

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If $F \sim_{GL_2(\mathbb{Z})} G$, then $F \sim_{\text{val}} G$. Hence

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The main question of today is: when is the inclusion (1) strict?

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Lemma

We have $\mathrm{Val}(F) = \mathrm{Val}(G)$, *but* $F \not\sim_{GL_2(\mathbb{Z})} G$ *by looking at discriminants. In particular,* $[F]_{GL_2(\mathbb{Z})} \subsetneq [F]_{\mathrm{val}}$.

Proof of lemma

Recall $F(X, Y) = X^3 - 3XY^2 - Y^3$, $R := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $F \circ R = F$ and $G(X, Y) := F(2X, Y)$. We must prove $\mathrm{Val}(F) = \mathrm{Val}(G)$.

Proof.

Clearly, $\mathrm{Val}(G) \subseteq \mathrm{Val}(F)$, so suffices to show $\mathrm{Val}(F) \subseteq \mathrm{Val}(G)$.

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$$z = F(x,y) = F(y, -x-y) = F(-x-y, x).$$

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Now at least one of $x, y, -x-y$ is even, say $x = 2m$. Then

$$z = F(x,y) = F(2m,y) = G(m,y),$$

so $z \in \mathrm{Val}(G)$, as desired. $\square$

Our main result

Theorem (K.–Fouvry)

Let $F \in \mathbb{Z}[X, Y]$ be a binary form of degree $d \geq 3$, and assume $\mathrm{disc}(F) \neq 0$. Then $[F]_{\mathsf{val}}$ consists of one or two $GL_2(\mathbb{Z})$–equivalence classes.

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Furthermore, in this case

$$[F]_{\mathrm{val}} = [G(X, Y)]_{GL_2(\mathbb{Z})} \cup [G(2X, Y)]_{GL_2(\mathbb{Z})}.$$

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- The possibilities for $\mathrm{Aut}(G)$ have been classified (as an abstract group). In particular, $|\mathrm{Aut}(G)| \leq 12$.
- Generically, we have $\mathrm{Aut}(F) = \{\mathrm{id}\}$ for d odd, $\mathrm{Aut}(F) = \{\mathrm{id}, -\mathrm{id}\}$ for d even. In particular, we generically have $[F]_{GL_2(\mathbb{Z})} = [F]_{\mathrm{val}}$.

High level proof strategy

Consider the surface $S \subseteq \mathbb{P}^3$ defined by

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However, the determinant method (developed by Heath-Brown and Salberger) shows that the rational points can only come in a rather structured way, namely from the lines on the surface.

The lines on the surface have been classified, which will then turn our problem into a question of lattice coverings.

Lattice coverings

Theorem (K.–Fouvry, "The lattice theorem")

Let F, G *with* $\mathrm{Val}(F) = \mathrm{Val}(G)$*, and let* $\rho \in GL_2(\mathbb{Q})$ *satisfy* $F = G \circ \rho$*.*

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$$\mathbb{Z}^2 = \bigcup_{\sigma_1 \in \mathrm{Aut}(F)} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 : \rho\sigma_1 \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \right\}$$

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Remark. Such a ρ must exist, since $\mathrm{Val}(F) = \mathrm{Val}(G)$ implies that S has many rational points, so by Step 1, 2, 3, there must be such a ρ.

The main result for trivial automorphism group

The "lattice theorem" is extremely useful. For example, if $\mathrm{Aut}(F) = \mathrm{id}$, we get

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This means precisely that $\rho \in GL_2(\mathbb{Z})$, so F and G are $GL_2(\mathbb{Z})$–equivalent.

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This argument also works if

$$\mathrm{Aut}(F) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} =: \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma \right\},$$

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However, if lattices $L_1, L_2 \subseteq \mathbb{Z}^2$ satisfy $L_1 \cup L_2 = \mathbb{Z}^2$, then $L_1 = \mathbb{Z}^2$ or $L_2 = \mathbb{Z}^2$. This still implies that F, G are $GL_2(\mathbb{Z})$–equivalent.

The general case

In general, we are led to the question: let $L_1, \ldots, L_6 \subseteq \mathbb{Z}^2$ be lattices. Suppose that $\mathbb{Z}^2 = L_1 \cup \cdots \cup L_6$. What can $L_1, \ldots, L_6$ be?

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- *There are exactly* 40 *coverings with* 6 *lattices.*

The cover with 3 lattices

The unique cover with 3 lattices is

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The other cases do not arise.

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