Extra Lecture Notes, SCI 113 Spring 2008
General Vector Spaces and Linear Transformations

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1. General Vector Spaces

1.1. Definition and Examples.

We have seen that a vector $\mathbf{v}$ in $\mathbb{R}^n$ is represented by a column matrix $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$. Also on $\mathbb{R}^n$ we have two operations (i) addition:

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix},$$

and (ii) scaler multiplication:

$$r\mathbf{u} = \begin{pmatrix} ru_1 \\ ru_2 \\ \vdots \\ ru_n \end{pmatrix}.$$

Furthermore these operations satisfy the following properties

(1) $\mathbf{u}+\mathbf{v}$ is in R^n whenever $\mathbf{u} \in R^n$ and $\mathbf{v} \in R^n$. (closed under addition)
(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ (addition is commutative)
(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$ (addition is associative)
(4) There is a vector called the zero vector, and denoted by $\mathbf{0}$ with the property that for every vector $\mathbf{u}$, one has $\mathbf{u}+\mathbf{0} = \mathbf{u}$. (additive identity)
(5) For every vector $\mathbf{u}$, there is a vector $-\mathbf{u}$ such that $\mathbf{u}+-\mathbf{u} = \mathbf{0}$. (additive inverse)
(6) $c\mathbf{u}$ is in R^n whenever c is a real number, and $\mathbf{u} \in \mathbb{R}^n$. (closed under scalar multiplication)
(7) $c(\mathbf{u}+\mathbf{v} = c\mathbf{u}+c\mathbf{v}$. (distributive property)
(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$. (distributive property)
(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$. (associative property)
(10) $1(\mathbf{u}) = \mathbf{u}$. (scalar identity)

Properties 1-10 allow us to generalize the notion of vector space in the following way.

Definition 1.1. *Let V be a set on which two operations* **vector addition** *and* **scalar multiplication** *are defined. If properties 1-10 above are satisfied, then V is called a vector space.*

Examples 1.1. (1) (The Vector Space of all 2×3 matrices) The set $M_{2,3}$ of all 2×3 matrices with the usual addition:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix},$$

and scalar multiplication

$$r\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \end{pmatrix}$$

is a vector space. It is easy to see that properties 1-10 are satisfied with the zero matrix playing the role of the additive identity, and $\begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{pmatrix}$ playing the role of the additive inverse of $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

(2) (The Vector Space of all $n \times m$ matrices) The set $M_{n,m}$ of all $n \times m$ matrices with the usual addition and scalar multiplication satisfy 1-10.

(3) (The Vector Space of all Polynomials of Degree less than or equal to two) Let P_2 be the set of all polynomials of the form

$$p(x) = a_2x^2 + a_1x + a_0,$$

where a_0, a_1, and a_2 are real numbers. Addition and scalar multiplication are defined as follows. The sum of two polynomials $p(x) = a_2x^2 + a_1x + a_0$, and $q(x) = b_2x^2 + b_1x + b_0$ is given by

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0).$$

If $p(x) = a_2x^2 + a_1x + a_0$ is a polynomial, and r is a real number, then the polynomial rp is given by

$$(rp)(x) = ra_2x^2 + ra_1x + ra_0.$$

It is easy to see that properties 1-10 are satisfied, with the zero polynomial $0(x) = 0$ playing the role of the additive identity, and $-p(x)$ playing the role of the additive inverse of $p(x)$.

(4) (The Vector space of Continuous Functions) Let C be the set of all real-valued continuous functions with the usual addition $(f+g)(x) = f(x)+g(x)$ and scalar multiplication $(rf)(x) = r(f(x))$. Since the sum of two continuous functions is continuous, and a multiple of a continuous function is continuous, we see that that properties (1) and (10) are satisfied. Furthermore, the zero function $0(x) = 0$ plays the role of the additive identity, and $-f(x)$ plays the role of the additive inverse of $f(x)$. Properties (2), (3), (7), 8, 9, and (10) follow from the usual properties of real numbers.

(5) Let $W = \{(x, y) \in R^2 : x + 2y = 0\}$. On W we consider the usual addition, and scalar multiplication. Note that if $(x, y), (u, v) \in W$, then $(x, y)+(u, v) = (x+u, y+v) \in W$, since $x+u+2(y+v) = (x+2y)+(u+2v) = 0$, and $r(x, y) = (rx, ry) \in W$ since $rx + 2ry = r(x + 2y) = 0$. Thus properties (1) and (6) are satisfied. The origin $(0, 0)$ is in W and is the additive identity. Also if $(x, y) \in W$, then $(-x, -y) \in W$ is the additive inverse of (x, y). Thus properties (4) and (5) are satisfied. The rest of the properties are easy to verify. Hence, W is a vector space.

1.2. **Spanning Sets, Linear Independence and Basis.**

Definition 1.2. *Let V be a vector space. A vector $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}$ if there exist scalars $c_1, c_2, \cdots, c_n$ such that*

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots c_n\mathbf{u_n}.$$

Examples 1.2. (1) Consider the vector space $\mathbb{R}^3$. The vector $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ is a linear combination of $\mathbf{u_1} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{u_2} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$ since $\mathbf{v} = 3u_2 + v_3$.

(2) Consider the vector space $M_{2,2}$ of all 2×2 matrices. Then the matrix (vector) $\mathbf{v} = \begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}$ is a linear combination of

$$\mathbf{v_1} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix},$$

since $\mathbf{v} = \mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3}$.

(3) Consider the vector space P_2 of polynomials of degree less than or equal to 2. The polynomial (vector) $\mathbf{v} = p(x) = 2 + 5x - x^2$ is a linear combination of $\mathbf{u_1} = p_1(x) = 1 + x - 2x^2$ and $\mathbf{u_2} = p_2(x) = x + x^2$ since $\mathbf{v} = 2\mathbf{u_1} + 3\mathbf{u_2}$.

Definition 1.3. *Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ a collection of vectors in V. We call S a spanning set of V if every vector $\mathbf{v}$ in V can be written as a linear combination of vectors in S.*

Examples 1.3. (1) Consider the standard basis $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$. The set $S = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a spanning set since if $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, then $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}$.

(2) The set $S = \{1, x, x^2\}$ is a spanning set for the vector space P_2 of all polynomials of degree less than or equal to 2.

(3) Let $\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. The set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a spanning set for $\mathbb{R}^3$. To see that, let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ be any vector. We want to find real numbers c_1, c_2, c_3 such that $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$. This leads to the following system of linear equations in the unknowns c_1, c_2, c_3 (here u_1, u_2, u_3 are considered as constants):

$$\begin{cases} c_1 - 2c_3 = u_1 \\ 2c_1 + c_2 = u_2 \\ 3c_1 + 2c_2 + c_3 = u_3. \end{cases}$$

The coefficient matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ has a non-zero determinant. Hence, A is invertible and the above system has a unique solution. Therefore, $\mathbf{u}$ can be written as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

Definition 1.4. *Let V be a vector space. A set $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ of vectors in V is said to be linearly independent if the vector equation*

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

Examples 1.4. (1) The vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent in $\mathbb{R}^3$. To see this, consider the equation $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} = \mathbf{0}$. This leads to the following system of linear equations (in the variables c_1, c_2, c_3

$$\begin{cases} c_1 - 2c_3 = 0 \\ 2c_1 + c_2 = 0 \\ 3c_1 + 2c_2 + c_3 = 0. \end{cases}$$

Using augmented matrices (Gauss elimination method), it is easy to see that the system has a unique solution $c_1 = c_2 = c_3 = 0$.

(2) Let $\mathbf{v_1} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. The set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent in $M_{2,2}$, the vector space of all 2×2 matrices (under the usual addition and scalar multiplication). To see this, suppose that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$. This leads to

$$\begin{cases} 2c_1 + 3c_2 + c_3 = 0 \\ c_1 = 0 \\ 2c_2 + 2c_3 = 0 \\ c_1 + c_2 = 0. \end{cases}$$

Using Gauss elimination method, it is easy to see that the system has a unique solution $c_1 = c_2 = c_3 = 0$.

(3) We show that the set $S = \{x^2+3x+1, 2x^2+x-1, 4x\}$ is linearly independent in P_2, the vector space of all polynomials of degree less than or equal to 2. Suppose that

$$c_1(x^2 + 3x + 1) + c_2(2x^2 + x - 1) + c_3(4x) = 0 = 0(x^2) + 0(x) + 0(1).$$

Rewriting, we get

$$(c_1 + 2c_2)x^2 + (3c_1 + c_2 + 4c_3)x + (c_1 - c_2) = 0.$$

This leads to the system

$$\begin{cases} c_1 + 2c_2 = 0 \\ 3c_1 + c_2 + 4c_3 = 0 \\ c_1 + c_2 = 0. \end{cases}$$

This system has a unique solution $c_1 = c_2 = c_3 = 0$. Hence, S is linearly independent.

1.3. **Basis and Dimension.**

Definition 1.5. *A set* $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ *in a vector space* V *is said to be a basis if*
1. S *is a spanning set for* V*.*
2. S *is linearly independent*

Examples 1.5. (1) Consider the vectors $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The set $S = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ forms a basis in $\mathbb{R}^3$ since the S is linearly independent (if $c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} = \mathbf{0}$, then $c_1 = c_2 = c_3 = 0$), and spans $\mathbb{R}^3$ (if $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, then $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}$). As we already know, the set S is called the standard basis in $\mathbb{R}^3$.

(2) Consider the set $S = \{\mathbf{u_1}, \mathbf{u_2}\}$, where $\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\mathbf{u_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The set S forms a basis. We first show linear independence. Suppose $c_1\mathbf{u_1} + c_2\mathbf{u_2} = \mathbf{0}$, this leads to the system

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0. \end{cases}$$

Hence, $c_1 = c_2 = 0$ and S is linearly independent. We now show that S is a spanning set. Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be any vector in $\mathbb{R}^2$, we want to find real numbers c_1, c_2 such that $\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2}$. This is equivalent to solving the following system

$$\begin{cases} c_1 + c_2 = v_1 \\ c_1 - c_2 = v_2. \end{cases}$$

Thus, $c_1 = \dfrac{v_1 + v_2}{2}$ and $c_2 = \dfrac{v_1 - v_2}{2}$. Hence, we have found c_1, c_2 such that $\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2}$. Thus S is a spanning set, and therefore S is a basis for $\mathbb{R}^2$.

(3) The set $S = \{1, x, x^2, x^3\}$ is a basis for P_3, the set of all polynomials of degree less than or equal to 3. Clearly, any polynomial $p \in P_3$ has the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, hence a linear combination of elements of S. This shows that S is a spanning set. Furthermore, S is linearly independent, since if $c_0 + c_1x + c_2x^2 + c_3x^3 = 0(x) = 0$, then $c_0 = c_1 = c_2 = c_3 = 0$. Thus, S is a basis.

(4) Consider $M_{2,2}$ the set of all 2×2 matrices. Let

$$\mathbf{v_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ forms a basis for $M_{2,2}$. Clearly, any matrix $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written as

$$\mathbf{u} = a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} + d\mathbf{v_4}.$$

hence, S is a spanning set. Furthermore, if $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + c_4\mathbf{v_4} = \mathbf{0}$ ($\mathbf{0}$ is the zero matrix), then $c_1 = c_2 = c_3 = c_4 = 0$, so that S is linearly independent, and therefore a basis.

2. Linear Transformations

2.1. **Definition and Examples.** Let V and W be vector spaces. A map $T : V \to W$ is a rule that assigns to each vector $\mathbf{v}$ of V a vector $\mathbf{w}$ of W denoted by $\mathbf{w} = T(\mathbf{v})$. The vector $\mathbf{w}$ is called the **image** of $\mathbf{v}$, and $\mathbf{v}$ is called the preimage of $\mathbf{w}$.

Definition 2.1. *Let V and W be vector spaces, and $T : V \to W$ a mapping. We call T a linear transformation if*

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v}$ in V and for all scalars a and b.

Examples 2.1.

(1) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \end{pmatrix}$. Notice that $T(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Thus,

$$\begin{aligned} T(a\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + b\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) &= A(a\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + b\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) \\ &= aA\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + bA\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= aT(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}) + bT(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}). \end{aligned}$$

So T is a linear transformation.

(2) In general if A is an $n \times n$ matrix, then the function $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{v}) = A\mathbf{v}$ defines a linear transformation.

(3) Let $M_{n,m}$ be the vector space of all $n \times m$ matrices. Define $T : M_{n,m} \to M_{n,m}$ by $T(A) = A^T$. By the properties of matrices we have

$$T(aA + bB) = (aA + bB)^T = (aA)^T + (bB)^T = aA^T + bB^T = aT(A) + bT(B).$$

Thus T is a linear transformation.

(4) Let C be the set of all real-valued continuous functions, and D the set of all differentiable functions with a continuous derivative. Both C and D are vector spaces under the usual addition and scalar multiplication of functions. Define $T : D \to C$ by $T(f) = f'$, where f' is the derivative of f (note that f' is an element of C). Then,

$$T(af + bg) = (af + bg)' = af' + bg' = aT(f) + bT(g).$$

Thus, T (i.e. the operation of taking derivatives) is a linear transformation.

(5) Let $P[c,d]$ be the vector space of all polynomials defined on the interval $[c,d]$. Define $T : P \to \mathbb{R}$ by $T(p) = \int_c^d p(t)dt$. Then,

$$T(ap+bq) = \int_c^d (ap(t)+bq(t)dt = a\int_c^d p(t)dt + b\int_c^d q(t)dt = aT(p)+bT(q).$$

Thus T (i.e. the operation of integration) is a linear transformation.

2.2. **Matrices for Linear Transformations.** Consider $\mathbb{R}^n$ with the standard basis $S = \{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$, so each $\mathbf{e_i}$ has n-coordinates each of which is 0 except for the ith coordinate which equals 1. Now let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. On $\mathbb{R}^m$ we also consider the standard basis $S' = \{\mathbf{e'_2}, \cdots, \mathbf{e'_m}\}$. Each $\mathbf{e'_i}$ has m coordinates each of which is zero except for the ith coordinate which equals 1. Suppose

$$T(\mathbf{e_1}) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, T(\mathbf{e_2}) = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, T(\mathbf{e_n}) = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Define an $m \times n$ matrix A as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We claim that $T(\mathbf{v}) = A\mathbf{v}$ for any vector $\mathbf{v} \in \mathbb{R}^n$. to see this, suppose

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + \cdots + v_n\mathbf{e_n}.$$

Since T is a linear transformation, then

$$\begin{aligned} T(\mathbf{v}) &= T(v_1\mathbf{e_1} + v_2\mathbf{e_2} + \cdots + v_n\mathbf{e_n}) \\ &= v_1T(\mathbf{e_1}) + v_2T(\mathbf{e_1}) + \cdots + v_nT(\mathbf{e_n}). \end{aligned}$$

On the other hand,

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\ &= \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{pmatrix} \\ &= v_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + v_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\ &= v_1 T(\mathbf{e_1}) + v_2 T(\mathbf{e_1}) + \cdots + v_n T(\mathbf{e_n}). \end{aligned}$$

Example 2.1. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 + v_2 - v_3 \\ -v_1 + 3v_2 - 2v_3 \\ 3v_2 + 4v_3 \end{pmatrix}.$$

To find the matrix A of T, we find the images of the standard basis:

$$T(\mathbf{e}_1) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, T(\mathbf{e}_2) = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, T(\mathbf{e}_3) = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}.$$

Thus, $A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{pmatrix}$, and $T(\mathbf{v}) = A\mathbf{v}$.

So far we have considered only the case when the vector space is $\mathbb{R}^n$ with the standard basis. Suppose now V is a vector space with (ordered) basis $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$, and W a vector space with (ordered) basis $B' = \{\mathbf{w_1}, \mathbf{w_2}, \cdots \mathbf{w_m}\}$. Let now $T : V \to W$ be a linear transformation such that

$$\begin{aligned} T(\mathbf{v_1}) &= a_{11}\mathbf{w_1} + a_{21}\mathbf{w_2} + \cdots a_{m1}\mathbf{w_m} \\ T(\mathbf{v_2}) &= a_{12}\mathbf{w_1} + a_{22}\mathbf{w_2} + \cdots a_{m2}\mathbf{w_m} \\ &\vdots \\ T(\mathbf{v_n}) &= a_{1n}\mathbf{w_1} + a_{2n}\mathbf{w_2} + \cdots a_{mn}\mathbf{w_m}. \end{aligned}$$

Define the $m \times n$ matrix A by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Note that the first column corresponds to the coefficients of $T(\mathbf{v_1})$ when expressed as a linear combination of the elements of the basis B', the second column corresponds to the coefficients of $T(\mathbf{v_2})$ when expressed as a linear combination of the elements of the basis B', $\cdots$, the nth column corresponds to the coefficients of $T(\mathbf{v_n})$ when expressed as a linear combination of the elements of the basis B'. The same proof as above (relative to the standard bases) shows that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v}$ in V (on the right hand side $\mathbf{v}$ must be expressed as a linear combination of elements of B, and then written as a column vector). The matrix A is called **the matrix of T relative to the bases B and B'.**

Example 2.2. Let P_1 be the vector space of all polynomials of degree less than or equal to 1, and P_2 the vector space of all polynomials of degree less than or equal to 2. Let $T : P_2 \to P_1$ be the differential operator, i.e. $T(p) = p'$. We want to find the matrix of T with respect to the bases $B = \{1, x, x^2\}$ on P_2, and $B' = \{1, x\}$ on P_1. We look at the images of the elements of B, and we write them as linear combinations of elements of B'.

$$\begin{aligned} T(1) &= 0 = 0(1) + 0(x) \\ T(x) &= 1 = 1(1) + 0(x) \\ T(x^2) &= 2x = 0(1) + 2(x). \end{aligned}$$

Hence, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

So, if $p(x) = a_0 + a_1 x + a_2 x^2$, then

$$T(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \end{pmatrix},$$

i.e. $T(p)(x) = a_1 + 2a_2 x$ as expected.

3. Exercises

(1) Show that the set M of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ is a vector space under the usual operation of addition and scalar multiplication.

(2) Show that the vector $\mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

(3) let $\mathbf{v_1} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$. Show that $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a spanning set for $\mathbb{R}^3$.

(4) Show that the set $S = \{x^2 - 1, 2x + 5\}$ is linearly independent in P_2, the vector space of all polynomials of degree at most 2.

(5) Show that the set $S = \{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\}$ forms a basis for $M_{2,2}$, the vector space of all 2×2 matrices.

(6) Suppose $T : M_{2,2} \to M_{2,2}$ is a linear transformation such that

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix},$$
$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)) = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix},$$
$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$
$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.$$

Find $T\left(\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\right)$.

(7) Find the standard matrix (i.e. relative to the standard bases) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 13v_1 - 9v_2 + 4v_3 \\ 6v_1 + 5v_2 - 3v_3 \end{pmatrix}.$$

(8) Let $B = \{1, x, x^2, x^3\}$ be a basis for P_3 (the vector space of polynomials of defree at most 3), and $B' = \{1, x, x^2, x^3, x^4\}$ a basis for P_4 (the vector space of polynomials of defree at most 4). Consider the linear transformation (defined on the basis vectors by)

$$T(x^k) = \int_0^x t^k \, dt.$$

Find the matrix of T relative to the bases B and B'.