Practice Final Measure and Integration 2014-15

(1) Let μ and ν be two measures on the measure space $(E,\mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$.
(a) Show that if f is any non-negative measurable function on $(E,\mathcal{B})$, then $\int_E f\,d\mu \leq \int_E f\,d\nu$.
(b) Prove that if ν is a finite measure, then $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\mu)$.

Proof (a) Suppose first that $f = 1_A$ is the indicator function of some set $A \in \mathcal{B}$. Then

$$\int_E f\,d\mu = \mu(A) \leq \nu(A) = \int_E f\,d\nu.$$

Suppose now that $f = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

$$\int_E f\,d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \leq \sum_{k=1}^{n} \alpha_k \nu(A_k) = \int_E f\,d\nu.$$

Finally, let f be a non-negative measurable function, then there exists a sequence of non-negative measurable step functions f_n such that $f_n \uparrow f$. By Beppo-Levi,

$$\int_E f\,d\mu = \lim_{n\to\infty} \int_E f_n\,d\mu \leq \lim_{n\to\infty} \int_E f_n\,d\nu = \int_E f\,d\nu.$$

Proof (b) From part (a) we see that if $f \in \mathcal{L}^1(\nu)$, then $f \in \mathcal{L}^1(\mu)$, i.e. $\mathcal{L}^1(\nu) \subseteq \mathcal{L}^1(\mu)$. If ν is a finite measure, then by Exercise 12.1 (ii) and the above, we have $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\nu) \subseteq L^1(\mu)$.

(2) Consider the measure space $((0,1],\mathcal{B}((0,1]),\lambda)$, where $\mathcal{B}((0,1])$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,1]$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,1]} e^{1/x}(1+n^2x)^{-1} \sin(ne^{-1/x}\,d\lambda(x).$$

Proof: Let $u_n(x) = e^{1/x}(1+n^2x)^{-1}\sin(ne^{-1/x}$, then $\lim_{n\to\infty} u_n(x) = 0$ for all $x \in (0,1]$. Since $|\sin y| \leq y$ for all $y \geq 0$, we have

$$|u_n(x)| \leq e^{1/x}(1+n^2x)^{-1}ne^{-1/x} = \frac{n}{1+n^2x} = \frac{1}{\sqrt{x}} \cdot \frac{n\sqrt{x}}{1+n^2x} \leq \frac{1}{\sqrt{x}}.$$

Since the function $\frac{1}{\sqrt{x}}$ is positive, measurable and the improper Riemann integrable on $(0,1]$ exists, it follows that it is Lebesgue integrable on $(0,1]$. By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \int_{(0,1]} e^{1/x}(1+n^2x)^{-1}\sin(ne^{-1/x}\,d\lambda(x) &= \lim_{n\to\infty} \int u_n(x)\,d\lambda(x) \\ &= \int \lim_{n\to\infty} u_n(x)\,d\lambda(x) = 0.\end{aligned}$$

(3) Let $(X,\mathcal{F},\mu)$ be a **finite** measure space. Assume $f \in \mathcal{L}^2(\mu)$ satisfies $0 < ||f||_2 < \infty$, and let $A = \{x \in X : f(x) \neq 0\}$. Show that

$$\mu(A) \geq \frac{(\int f\,d\mu)^2}{\int f^2\,d\mu}.$$

Solution: Since $f = 0$ on A^c, we have $\int f\,d\mu = \int f\mathbf{1}_A\,d\mu$. Since μ is a finite measure and $(\mathbf{1}_A)^2 = \mathbf{1}_A$, then

$$||\mathbf{1}_A||_2 = (\mu(A))^{1/2} < \infty.$$

Thus, $\mathbf{1}_A \in \mathcal{L}^2(\mu)$ and by Hölder's inequality

$$\int f\,d\mu \leq ||f||_2||\mathbf{1}_A||_2 = |f||_2(\mu(A))^{1/2}.$$

Squaring both sides and dividing by

$$||f||_2^2 = \int f^2\,d\mu\,(>0),$$

we get

$$\mu(A) \geq \frac{(\int f\,d\mu)^2}{\int f^2\,d\mu}.$$

bigskip

(4) Let $1 \leq p < \infty$, and suppose $(X, \mathcal{A}, \mu)$ is a finite measure space. Let $(f_n)_n \in \mathcal{L}^p(\mu)$ be a sequence converging to f in μ measure.

(a) Show that

$$\int |f|^p\,d\mu \leq \liminf_{n\to\infty} \int |f_n|^p\,d\mu.$$

(b) Show that $\lim_{n\to\infty} n^p\mu(\{|f| > n\}) = 0$.

Solution (a): By definition of the liminf, we can find a subsequence $(f_{n(j)})_j$ such that

$$\lim_{j\to\infty} \int |f_{n(j)}|^p\,d\mu = \liminf_{n\to\infty} \int |f_n|^p\,d\mu.$$

By Exercise 16.10(iii), the sequence $(|f_n|^p)$ converges in μ measure to $|f|^p$. By Exercise 16.10(ii) applied to the sequence $(|f_n|^p)$ and the fact that $\mu(X) < \infty$, there exists a subsequence $(f_{m(j)})$ of $(f_{n(j)})_j$ such that

(i) $(|f_{m(j)}|^p)$ converges μ a.e. to $|f|^p$, and

(ii) $\lim_{j\to\infty} \int |f_{m(j)}|^p\,d\mu = \liminf_{n\to\infty} \int |f_n|^p\,d\mu$.

By Fatou's Lemma

$$\int |f|^p\,d\mu = \int \liminf_{j\to\infty} |f_{m(j)}|^p\,d\mu \leq \liminf_{j\to\infty} \int |f_{m(j)}|^p\,d\mu = \lim_{j\to\infty} \int |f_{m(j)}|^p\,d\mu = \liminf_{n\to\infty} \int |f_n|^p\,d\mu.$$

Solution (b): Note that $f \in \mathcal{L}^p(\mu)$ and hence by Corollary 10.13,

$$\mu(\{|f|^p = \infty\}) = \mu(\{|f| = \infty\}) = 0.$$

Thus,

$$\lim_{n\to\infty} |f|^p\mathbf{1}_{\{|f|>n\}} = |f|^p\mathbf{1}_{\{|f|=\infty\}} = 0 \;\; \mu \; a.e.$$

Since for each n, $|f|^p\mathbf{1}_{\{|f|>n\}} \leq |f|^p$ and $|f|^p \in \mathcal{L}^1(\mu)$, we have by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int |f|^p\mathbf{1}_{\{|f|>n\}}\,d\mu = 0.$$

Now,

$$n^p\mu(\{|f| > n\}) = \int n^p\mathbf{1}_{\{|f|>n\}}\,d\mu \leq \int |f|^p\mathbf{1}_{\{|f|>n\}}\,d\mu,$$

and from the above we get $\lim_{n\to\infty} n^p\mu(\{|f| > n\}) = 0$.

(5) Let $E = \{(x, y) : y < x < 1, \, , 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = x^{-3/2}\cos(\frac{\pi y}{2x})$.

(a) Show that f is $\lambda \times \lambda$ integrable on E.

(b) Define $F : (0, 1) \to \mathbb{R}$ by $F(y) = \int_{(y,1)} x^{-3/2}\cos(\frac{\pi y}{2x})\,d\lambda(x)$. Determine the value of

$$\int F(y)\,d\lambda(y).$$

Solution (a) : Notice that f is continuous, and hence measurable. Furthermore, $|f(x,y)| \le x^{-3/2}$. The function $g(x,y) = x^{-3/2}$ is non-negative and measurable on E, hence by Tonelli's Theorem,

$$\begin{aligned} \int_E |f(x,y)|\, d(\lambda\times\lambda)(x,y) &\le \int_E g(x,y)\, d(\lambda\times\lambda)(x,y) \\ &= \int_0^1 \int_0^x x^{-3/2}\, dy\, dx \\ &= \int_0^1 x^{-1/2}\, dx = 2. \end{aligned}$$

Notice that the integrands are Riemann integrable, hence the Riemann integral equals the Lebesgue integral. This shows that f is $\lambda\times\lambda$ integrable on E.

Solution (b) : By Fubini's Theorem

$$\int\int f(x,y)\, d\lambda(x)\, d\lambda(y) = \int\int f(x,y)\, d\lambda(y)\, d\lambda(x).$$

Notice that for each fixed $0 < x < 1$, the function $f(x,y)$ is Riemann-integrable in y on the interval $(0,x)$ and

$$\int_0^x x^{-3/2} \cos(\frac{\pi y}{2x})\, dy = \frac{2}{\pi}\, x^{-1/2},$$

and the function $\frac{2}{\pi}x^{-1/2}$ is Riemann-integrable in x on the interval $(0,1)$, and

$$\int_0^1 \frac{2}{\pi}\, x^{-1/2}\, dx = \frac{4}{\pi}.$$

Thus,

$$\int F(y)\, d\lambda(y) = \int\int f(x,y)\, d\lambda(x)\, d\lambda(y) = \int_0^1 \int_0^x x^{-3/2} \cos(\frac{\pi y}{2x})\, dy\, dx = \frac{4}{\pi}.$$

(6) Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$) and let $\{f_n\}$ be a sequence in $\mathcal{L}^1(\mu)$ such that $\int_X |f_n| d\mu = n$ for all $n \ge 1$. Let

$$A_n = \{x : |f_n(x) - \int_X f_n d\mu| \ge n^3\}.$$

(a) Show that $\mu\left(\bigcap_{m\ge1}\bigcup_{n\ge m} A_n\right) = 0$.
(b) Use part (a) to show that for every $\epsilon > 0$ there exists $m_0 \ge 1$ such that

$$\mu\{x \in X : |f_n(x)| < n^3 + n,\ \text{for all } n \ge m_0\} > 1 - \epsilon.$$

Proof (a) By Markov Inequality we have

$$\mu(A_n) \le \frac{1}{n^3}\int_X |f_n(x) - \int_X f_n d\mu|\, d\mu \le \frac{2n}{n^3} = \frac{2}{n^2}.$$

Since $\displaystyle\sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \frac{2}{n^2} < \infty$, it follows by Borel-Cantelli Lemma (Exercise 6.9) that

$$\mu\left(\bigcap_{m\ge1}\bigcup_{n\ge m} A_n\right) = 0.$$

Proof (b) By part (a) we have $\mu\left(\bigcup_{m\ge1}\bigcap_{n\ge m} A_n^c\right) = 1$. By Theorem 4.4(iii),

$$\lim_{m\to\infty} \mu\left(\bigcap_{n\ge m} A_n^c\right) = \mu\left(\bigcup_{m\ge1}\bigcap_{n\ge m} A_n^c\right) = 1.$$

Hence, given $\epsilon > 0$ there exists $m_0 \geq 1$ such that $\mu\left(\bigcap_{n\geq m_0} A_n^c\right) > 1-\epsilon$. But for $x \in \bigcap_{n\geq m_0} A_n^c$ one has for $n \geq m_0$,

$$|f_n(x)| - |\int f_n\,d\mu| \leq |f_n(x) - \int f_n(x)\,d\mu| < n^3,$$

and thus, $|f_n(x)| < n^3+n$. This implies that

$$\mu\{x \in X : |f_n(x)| < n^3+n, \text{ for all } n \geq m_0\} \geq \mu\left(\bigcap_{n\geq m_0} A_n^c\right) > 1-\epsilon.$$

(7) Suppose μ and ν are finite measures on the measurable space $(X, \mathcal{A})$ which have the same null sets. Show that there exists a measurable function f such that $0 < f < \infty$ μ a.e. and ν a.e. and for all $A \in \mathcal{A}$ one has

$$\nu(A) = \int_A f\,d\mu \quad \text{and} \quad \mu(A) = \int_A \frac{1}{f}\,d\nu.$$

Proof The fact that μ and ν have the same null sets implies that $\nu \ll \mu$ and $\mu \ll \nu$ (in fact in this case we refer to μ and ν as equivalent measure). So the notions μ a.e. and ν a.e. are the same. By Radon-Nikodym Theorem there exist $f \in \mathcal{L}^1_+(\mu)$ and $g \in \mathcal{L}^1_+(\nu)$ such that for all $A \in \mathcal{A}$,

$$\nu(A) = \int_A f\,d\mu \quad \text{and} \; \mu(A) = \int_A g\,d\nu.$$

Furthermore, the functions f and g are unique μ and ν a.e. By Exercise 1 of the last set of exercises exercisesRadonNikodym.pdf, we have for any $A \in \mathcal{A}$,

$$\nu(A) = \int_A 1\,d\nu = \int_A f\,d\mu = \int_A fg\,d\nu.$$

By Corollary 10.14(i) this implies that $1 = fg$ ν and hence μ a.e. From this and Corollary 10.13 we conclude that $0 < f < \infty$ and $g = 1/f$ μ and ν a.e.

(8) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f_n, f \in \mathcal{M}(\mathcal{A})$, $n \geq 1$. Show that f_n converges to f in μ measure **if and only if** $\lim_{n\to\infty} \int \frac{|f_n - f|}{1+|f_n-f|}\,d\mu = 0$.

Solution: First note that $\frac{|f_n - f|}{1+|f_n-f|} \leq 1$ for all $n \geq 1$, and since $\mu(X) < \infty$ we have $1 \in \mathcal{L}^1(\mu)$.

Now assume that $f_n \xrightarrow{\mu} f$, and let $\epsilon, \delta > 0$, then there exists N such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \delta\}) < \epsilon, \quad \text{for all } n \geq N.$$

Let $A = \{x \in X : |f_n(x) - f(x)| > \delta\}$, then for all $n \geq N$

$$\int \frac{|f_n - f|}{1+|f_n-f|}\,d\mu = \int_A \frac{|f_n - f|}{1+|f_n-f|}\,d\mu + \int_{A^c} \frac{|f_n - f|}{1+|f_n-f|}\,d\mu \leq \int_A 1\,d\mu + \int_{A^c} \delta\,d\mu.$$

Thus, for all $n \geq N$

$$\int \frac{|f_n - f|}{1+|f_n-f|}\,d\mu \leq \epsilon + \delta\mu(X).$$

Thus, $\lim_{n\to\infty} \int \frac{|f_n - f|}{1+|f_n-f|}\,d\mu = 0$.

Conversely, assume $\lim_{n\to\infty} \int \frac{|f_n - f|}{1+|f_n-f|}\,d\mu = 0$, and let $\epsilon > 0$. There exists N such that

$$\int \frac{|f_n - f|}{1+|f_n-f|}\,d\mu < \epsilon^2/(1+\epsilon), \quad \text{for all } n \geq N.$$

Observe first that

$$|f_n - f| > \epsilon \iff \frac{|f_n - f|}{1+|f_n-f|} > \frac{\epsilon}{1+\epsilon}.$$

Thus, by Markov Inequality, we have for all $n \geq N$

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = \mu(\{x \in X : \frac{|f_n - f|}{1 + |f_n - f|} > \frac{\epsilon}{1+\epsilon}\}) \leq \frac{1+\epsilon}{\epsilon} \int \frac{|f_n - f|}{1 + |f_n - f|} \, d\mu < \epsilon.$$

Thus, $f_n \xrightarrow{\mu} f$.