Solutions Book Chapter 28, SCI 113 Spring 2008

(1) **Exercise 28.3 (a)** $f(x,y) = 3x + 7y - 2$, $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = 3$ (answer is independent of x and y, hence, $\frac{\partial f}{\partial x}(2,1) = f_x(2,1) = 3$. Similarly, $\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = 7$, so that $\frac{\partial f}{\partial y}(2,1) = f_y(2,1) = 7$. **(c)** $f(x,y) = 2x^2 - 3y^2 - 2xy - x - y + 1$, $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = 4x - 2y - 1$, so $\frac{\partial f}{\partial x}(2,1) = f_x(2,1) = 5$. $\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = -6y - 2x - 1$, so $\frac{\partial f}{\partial y}(2,1) = f_y(2,1) = -11$. **(f)** $f(x,y) = (x-1)(y-2)$, $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = (y-2)$, and $\frac{\partial f}{\partial x}(2,1) = f_x(2,1) = -1$. $\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = (x-1)$, and $\frac{\partial f}{\partial y}(2,1) = f_y(2,1) = 1$. **(e)** $f(x,y) = \frac{1}{xy}$, $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \frac{-1}{x^2 y}$, and $\frac{\partial f}{\partial x}(2,1) = f_x(2,1) = \frac{-1}{4}$. $\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \frac{-1}{xy^2}$, and $\frac{\partial f}{\partial y}(2,1) = f_y(2,1) = \frac{-1}{2}$.

(2) **Exercise 28.4 (a)** Let $u = ax + by$, then $z = g(u)$. By the chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = g'(u)a = ag'(ax+by),$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = g'(u)b = bg'(ax+by),$$

If $z = g(u) = \cos u$, then $\frac{\partial z}{\partial x} = -a\sin(ax+by)$, and $\frac{\partial z}{\partial y} = -b\sin(ax+by)$.
If $z = g(u) = e^u$, then $\frac{\partial z}{\partial x} = ae^{ax+by}$, and $\frac{\partial z}{\partial y} = be^{ax+by}$.

(b) If $z = g(\sin xy)$, then $u = \sin xy$. By the chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = g'(u)y\cos xy = y\cos xy g'(\sin xy),$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = g'(u)x\cos xy = x\cos xy g'(\sin xy).$$

If $g(u) = e^u$, then $z = e^{\sin xy}$. Applying the above formulas, or by differentiating directly, we get $\frac{\partial z}{\partial x} = y\cos xy\, e^{\sin xy}$ and $\frac{\partial z}{\partial y} = x\cos xy\, e^{\sin xy}$.

(c) $V = g(r)$, where $r = \sqrt{x^2+y^2}$. Recall theat in polar coordinates $x = r\cos\theta$, and $y = r\sin\theta$, where θ is the angle between the x-axis and the line joining the origin to the point (x,y). By the chain rule

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} = g'(r)\frac{x}{\sqrt{x^2+y^2}} = g'(\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}},$$

and

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} = g'(r)\frac{y}{\sqrt{x^2+y^2}} = g'(\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}}.$$

We now express $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ in terms of r and θ:

$$\frac{\partial V}{\partial x} = g'(\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}} = g'(r)\cos\theta,$$
$$\frac{\partial V}{\partial y} = g'(\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}} = g'(r)\sin\theta.$$