Assignment 2, SCI 113 Spring 2008
due date: March 20,2008

- The assignment should be handed in on paper, and not by email. Do not forget to put your name on it.
- Each student should hand in his/her own assignment. It is not allowed to hand in assignments with joint authorship.
- Do not just give the final solution to a problem. Provide full argumentation in clear sentences. The argumentation should clarify the steps followed in your reasoning and calculations.

(1) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, and $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are vectors in the plane $\mathbb{R}^2$ such that

$$A\mathbf{u} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \text{ and } A\mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

(a) Determine the elements of the matrix A, i.e. find a, b, c, d.

(b) Does the inverse matrix A^{-1} exist? If yes, find A^{-1}.

(c) Consider the map T defined on the plane $\mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. Show that the image of the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ is a parallelogram. Show that the area of this parallelogram is $\det(A)$.

(2) Let Δ be the triangle in the plane with vertices the points (x_1, y_1), (x_2, y_2) and (x_3, y_3). Define the matrix A by

$$A = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

Show that the area of triangle Δ is equal to $\pm\frac{\det(A)}{2}$, where $\pm$ sign is chosen to give a positive area.

(3) Let

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of A and their corresponding eigenvectors.

(b) Show that A is diagonalizable, and find a formula for A^n for any positive integer n.

(4) Do problem 8.15 on p.174 of the textbook Mathematical Techniques by D.W. Jordan and P. Smith.

(5) In this problem, you will be asked to decode a given cryptogram. A cryptogram is a message written according to a secret code. One way to code and decode a message is by means of matrix multiplication. To do this you first assign a number to each letter of the alphabet (with 0 assigned a blank space) as follows: $0 = -$, $1 = A$, $2 = B$, $3 = C$, $4 = D$, $5 = E$, $6 = F$, $7 = G$, $8 = H$, $9 = I$, $10 = J$, $11 = K$, $12 = L$, $13 = M$, $14 = N$, $15 = O$, $16 =$

P, $17 = Q$, $18 = R$, $19 = S$, $20 = T$, $21 = U$, $22 = V$, $23 = W$, $24 = X$, $25 = Y$, $26 = Z$.
Then, the message is converted to numbers and divided into blocks of length 3, called **uncoded row matrices** each having 3 entries. For example, the message **MEET ME MONDAY** has the following uncoded row matrices (written one after the other):

$$(13\ 5\ 5)\,(20\ 0\ 13)\,(5\ 0\ 13)\,(15\ 14\ 4)\,(1\ 25\ 0)$$
$$(M\,E\,E)\,(T - M)\,(E - M)\,(O\,N\,D)\,(A\,Y -).$$

To **encode** the message one chooses a 3×3 non-singular matrix, called the **encoding matrix**, and then multiplies (from the left) each uncoded row matrix by A. So, for example if $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$, then the coded message of **MEET ME MONDAY** is obtained as follows:

$$(13\ 5\ 5)A = (13\ \ -26\ 21)$$
$$(20\ 0\ 13)A = (33\ \ -53\ \ -12)$$
$$(5\ 0\ 13)A = (18\ \ -23\ \ -42)$$
$$(15\ 14\ 4)A = (5\ \ -20\ 56)$$
$$(1\ 25\ 0)A = (-24\ \ -23\ 7).$$

So the coded row matrices are now

$$(13\ \ -26\ 21)(33\ \ -53\ \ -12)(18\ \ -23\ \ -42)(5\ \ -20\ 56)(-24\ \ -23\ 7).$$

Removing the brackets produces the following cryptogram

$$13\ \ -26\ 21\ 33\ \ -53\ \ -12\ 18\ \ -23\ \ -42\ 5\ \ -20\ 5\ \ -24\ \ -23\ 7.$$

If somebody gives you the above cryptogram, and if you know the matrix A, then you can decode the message by multiplying each coded row matrix (from the left) by

$$A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

to get the original uncoded row matrices, and then change the numbers to letters in order to read the original message. For example $(13\ -26\ 21)A^{-1} = (13\ 5\ 5)$, if we now replace the numbers $13, 5, 5$ by the corresponding letters we get MEE, and so on for the other remaining row matrices.

Now do the following problem. Suppose that the encoding matrix is

$$A = \begin{pmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{pmatrix},$$

and you receive the the following cryptogram

$$33\ 9\ 9\ \ 55\ \ 28\ \ 14\ \ 95\ \ 50\ \ 25\ \ 99\ \ 53\ \ 29\ \ -22\ \ -32\ \ -9.$$

Decode this cryptogram.