Assignment 4, SCI 113 Spring 2008
due date: May 7,2008

- Write your name and student number on each page you hand in.
- You are allowed to use the book Mathematical Techniques by Jordan and Smith and the lecture notes by Frits Beukers.
- You should explain how you have calculated your answers
- You should do this assignment individually. You are not allowed to work in groups.

(1) (a) Calculate the Taylor polynomial of degree 4 around the point $\pi/6$ for the function $f(x) = \sin^2 x$.

(b) If you use the polynomial of part (a) to approximate $\sin 31°$, how many correct digits do you expect?

(c) Use the polynomial of part (a) to approximate $\sin 31°$.

(2) Using Taylor series, find the following limits

(a) $\lim_{x \to 0} \frac{x \sin x}{\cos x - 1}$.

(b) $\lim_{x \to 1} \frac{\ln x}{x^2 - 1}$.

(3) (a) Show that $10! \geq 3 \cdot 10^6$. Here you may use your calculator.

(b) Show that

$$n! \geq 3 \cdot 10^{n-4}$$

for all integers $n \geq 10$.

(c) Show that for any $n \geq 9$,

$$|e^{-1} - \sum_{k=0}^{n} \frac{(-1)^k}{k!}| \leq 10^{3-n}.$$

(d) We want to approximate e^{-1} by means of the finite sum $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$. How large should n be in order to approximate e^{-1} with an accuracy of 15 decimals? Motivate your answer.

(e) Use Mathematica to make a table which confirms the correctness of your answer.

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $z = f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

(a) Find all stationary points of the above function.

(b) For each stationary point obtained in part (a), determine whether f attains a local maxima, local minima, a saddle point, or is of a different nature.

(c) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(2, 1, 11)$.

(5) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with the property that the partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$$

are all defined and continuous. For x, y different from zero, we consider the function

$$Q(x,y) = \frac{f(x,y) - f(0,y) - f(x,0) + f(0,0)}{xy}.$$

(a) Show that for each x, y there exists a real number $t_{x,y}$ lying between 0 and x such that

$$Q(x,y) = \frac{\frac{\partial f}{\partial x}(t_{x,y}, y) - \frac{\partial f}{\partial x}(t_{x,y}, 0)}{y}.$$

Hint: apply the mean value theorem to the one variable function $\phi(x) = f(x,y) - f(x,0)$.

(b) Show that in addition there exists a real number $s_{x,y}$ lying between 0 and y such that

$$Q(x,y) = \frac{\partial^2 f}{\partial y \partial x}(t_{x,y}, s_{x,y}).$$

(c) Show that

$$\lim_{(x,y)\to(0,0)} Q(x,y) = \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

(d) By applying a similar argument as above, show that also

$$\lim_{(x,y)\to(0,0)} Q(x,y) = \frac{\partial^2 f}{\partial x \partial y}(0,0).$$