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Measure and Integration Exercises 11

1. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$. Suppose $f, g : E \to \mathbb{R}$ are measurable functions such that $f_n \to f$ in μ-measure and $f_n \to g$ μ a.e. Show that $f = g$ μ a.e.

2. Consider the measure space $([a, b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a, b]$, and λ is the restriction of the Lebesgue measure on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be any continuous function. Show that the Riemann integral of f on $[a, b]$ is equal to the Lebesgue integral of f on $[a, b]$, i.e.

$$(R) \int_a^b f(x)dx = \int_{[a,b]} f d\lambda.$$

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$.
   (a) Let $f : E \to \mathbb{R}$ be a measurable function such that $\sum_{n=0}^{\infty} \mu(|f - f_n| \geq \epsilon)) < \infty$ for all $\epsilon > 0$. Show that $f_n \to f$ in μ-measure and μ a.e.
   (b) Let (ϵ_n) be a sequence of positive real numbers such that $\sum_n \epsilon_n < \infty$. Prove that if $\sum_{n=0}^{\infty} \mu(|f_{n+1} - f_n| \geq \epsilon_n)) < \infty$, then there exists a measurable function $g : E \to R$ such that $f_n \to g$ in μ-measure and μ a.e.

4. Let f and $\{f_n\}$ be measurable real valued functions on a measure space $(E, \mathcal{B}, \mu)$ such that $f_n \to f$ in μ-measure, and $\sup_{n\geq 1} ||f_n||_{L^1(\mu)} < \infty$. Show that f is μ-integrable, and

$$\lim_{n\to\infty} \big| ||f_n||_{L^1(\mu)} - ||f||_{L^1(\mu)} - ||f_n - f||_{L^1(\mu)} \big| = |||f_n| - |f| - |f_n - f|||_{L^1(\mu)} = 0.$$

   Conclude that if $||f_n||_{L^1(\mu)} \to ||f||_{L^1(\mu)} \in \mathbb{R}$, then $||f_n - f||_{L^1(\mu)} \to 0$.