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Measure and Integration 2006-Selected Solutions Chapters 11/13

1. (**Exercise 11.7 p. 101**) Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and suppose $u \in \mathcal{L}^1(\mu)$. Define a function $I : (0, \infty) \to R$ by $I(x) = \int_{(0,x)} u(t)\, d\mu(t) = \int \mathbf{1}_{(0,x)}(t) u(t)\, d\mu(t)$. Show that if μ has no atoms (i.e. $\mu(\{x\}) = 0$ for all x), then I is continuous.

2. (**Extra exercise 1**) Consider the measure space $([a, b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a, b]$, andλ is the restriction of the Lebesgue measure on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function. Prove that f is continuous λ a.e.

3. (**Extra exercise 2**) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measure space, and let $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ be the corresponding product measurable space, where $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{A} \times \mathcal{B})$.

   (a) Show that for all $E \in \mathcal{A} \otimes \mathcal{B}$, and for all $x_0 \in X$ and all $y_0 \in Y$, one has $E_{x_0} = \{y \in Y : (x_0, y) \in E\} \in \mathcal{B}$ and $E_{y_0} = \{x \in X : (x, y_0) \in E\} \in \mathcal{A}$.

   (b) Let $f : X \times Y \to \overline{\mathbb{R}}$ be $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\overline{\mathbb{R}})$ measurable. Show that for all $x_0 \in X$ and all $y_0 \in Y$, the functions $f_{x_0} : Y \to \overline{\mathbb{R}}$ and $f_{y_0} : X \to \overline{\mathbb{R}}$ given by $f_{x_0}(y) = f(x_0, y)$ and $f_{y_0}(x) = f(x, y_0)$ are $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$ respectively $\mathcal{B}/\mathcal{B}(\overline{\mathbb{R}})$ measurable.

   (c) Assume that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are σ-finite. Let $E \in \mathcal{A} \otimes \mathcal{B}$. Show (using Theorem 13.5) that the following are equivalent:

      (i) $(\mu \otimes \nu)(E) = 0$.

      (ii) $\mu(E_y) = 0$ for ν almost every y,

      (iii) $\nu(E_x)) = 0$ for μ almost every x.