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Measure and Integration extra problems

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$ and λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2^{-k} & \text{if } x \in [k, k+1),\ k \in \mathbb{Z},\ k \geq 0. \end{cases}$$
   (a) Show that f is measurable, i.e. $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$.
   (b) Determine the values of $\lambda(\{f > 1\})$, $\lambda(\{f < 1\}$ and $\lambda(\{1/4 \leq f < 1\}$.
   (c) Determine the value of $\int f \, d\mu$.

2. Let $(X, \mathcal{B}, \mu)$ be a measure space, and $(A_n)_n \subset \mathcal{B}$ such that $\mu(A_n \cap A_m) = 0$ for $m \neq n$. Show that $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$.

3. Let $(X, \mathcal{B}, \nu)$ be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n} \nu(B \cap E_n)/(\nu(E_n) + 1)$.
   (a) Prove that μ is a finite measure on $(X, \mathcal{B})$.
   (b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ **if and only if** $\nu(B) = 0$.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\overline{\mathcal{B}}^{\mu}$ be the completion of the σ-algebra $\mathcal{B}$ with respect to the measure μ (see exercise 4.13, p.29). We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{B}}^{\mu}$. Suppose $f : E \to E$ is a function such that $f^{-1}(B) \in \mathcal{B}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{B}}^{\mu}$.

5. Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra over X containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.
   (a) Show that $\mathcal{D}$ is a σ-algebra over X.
   (b) Show that $\mathcal{D} = \sigma(\mathcal{C})$.

6. Let $(X, \mathcal{B}, \mu)$ be a probability space, i.e. $\mu(X) = 1$. Let $f : X \to [0, 1)$ be a measurable function such that $\mu\left(f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n})\right) = \frac{1}{2^n}$ for $n \geq 1$ and $k = 0, 1, \cdots, 2^n - 1$. Show that $\int_X f^2 \, d\mu = \frac{1}{3}$.