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# Mastermath Course: Ergodic Theory

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> General information

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All the information on the next slides can (in time) also be found on the **website**:

<http://www.staff.science.uu.nl/~kraai101/MasterMath2014.html>

On the website there is a link to the **lecture notes**. There are also references to other textbooks.

There are three hours scheduled each week. Most of the time we will use only **two hours**. Every now and then we will need some extra time and have a longer lecture.

An overview of the **material treated** will appear on the website.



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- Homework exercises: 20%
- Take home exam: 40%
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# Homework

There will be **four** homework sets.

The deadlines for the homework are 2 p.m. on

- September 30
- October 21
- November 25
- December 16

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## Take home exam

We will give you the exercises for the take home exam on October 28.

On the 4th of November there is no class.

The deadline for handing in the take home exam is at 2 p.m. on November 11.

You are **not** allowed to work together.



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## Final exam

The final exam will be a written three hour test in January. The exact date and time still need to be fixed.



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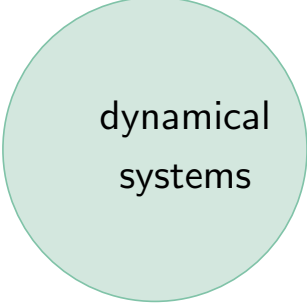


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> What is ergodic theory?

## Many different fields

Ergodic theory uses techniques and concepts from many different fields of mathematics.



dynamical  
systems



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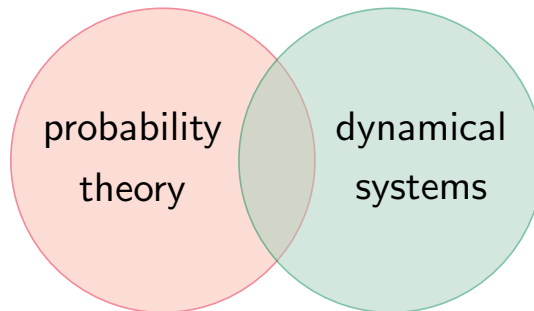


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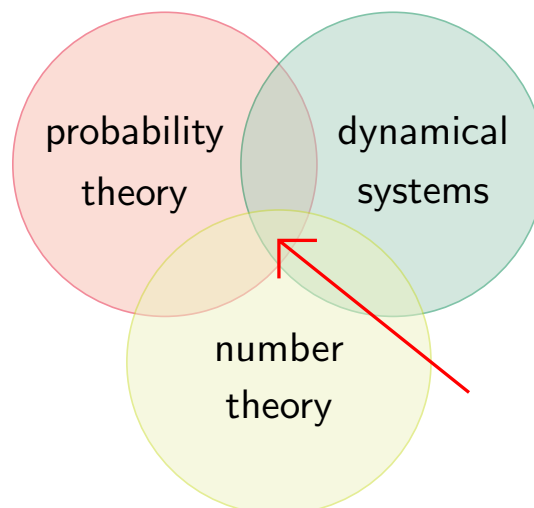


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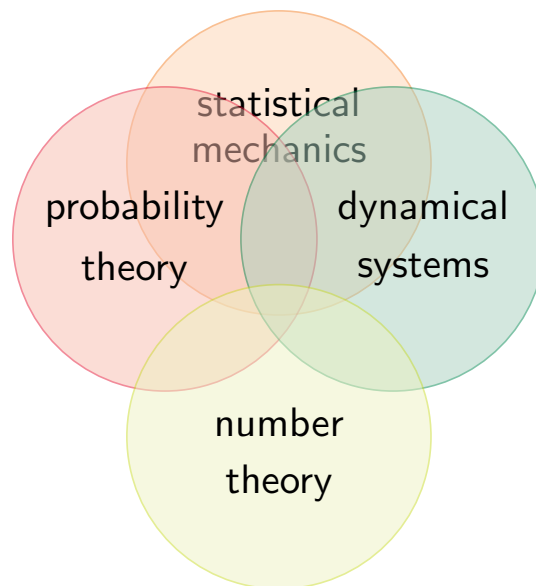


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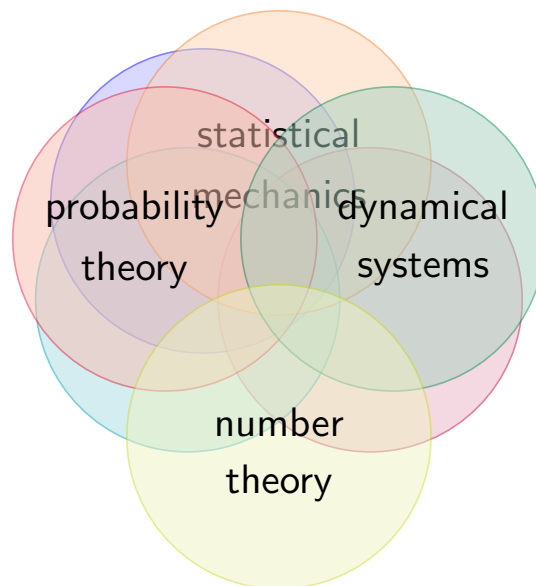


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> What is ergodic theory?

## Boltzmann's hypothesis

The origins of ergodic theory date back to the work on statistical mechanics by Boltzmann.

Ludwig Boltzmann (1844-1906) was an Austrian physicist and is seen as the founder of statistical mechanics.



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> What is ergodic theory?

## What's in a name

Boltzmann was the first to use the word *ergodic*.

It is a combination of two greek words: *ergon* = work and *odos* = path. These words are related to the conjecture that lies at the basis of the field.



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> Historical perspective

## Boltzmann's hypothesis

Boltzmann conjectured that:

*For large systems of interacting particles in equilibrium, the time average along a single trajectory equals the space average.*

The hypothesis as he stated it was wrong and ergodic theory originated from the search for conditions under which it is true.



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> Historical perspective

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> Historical perspective

## Ergodic Theorems

Conditions under which Boltzmann's hypothesis is true are given in the **Ergodic Theorems**. We will see two of these theorems in this course:

The Birkhoff Ergodic Theorem and the Mean (or Von Neumann) Ergodic Theorem.



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> Historical perspective

## Birkhoff Ergodic Theorem

Birkhoff was the first to publish his Ergodic Theorem.

George Birkhoff (1884-1944) was an American mathematician. He was an editor for the journal to which Von Neumann sent his results and he allegedly stalled the publication process, so that his result was published first.



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> Historical perspective

## Mean Ergodic Theorem

Soon after the publication of Birkhoff's result, the Ergodic Theorem by Von Neumann was published.

John Von Neumann (1903-1944) was a Hungarian mathematician who lived and worked in the United States for a large part of his life. He was actively involved in the development of the atomic bomb.



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> Historical perspective

## In the mean time ...

At the same time Poincaré was working on important results regarding recurrence.

Henri Poincaré (1854-1912) was a French mathematician who did very many important things.



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> Historical perspective

## Poincaré Recurrence Theorem

The **Poincaré Recurrence Theorem** will be one of the first results that we will state and prove.



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> Historical perspective

## Classification of systems

Another important theme of this course is the **classification of systems** according to their ergodic properties.

Concepts such as **isomorphism** and **entropy** play an important role here.



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> Historical perspective

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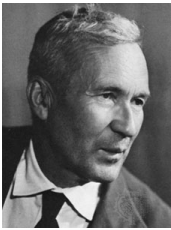
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> Historical perspective

# Entropy

Entropy reflects the amount of **randomness** present in a system.

People involved here are, in alphabetical ordering:



Andrey Kolmogorov  
(1903 - 1987)



Donald Ornstein  
(1934 - )



Yakov Sinai  
(1935 - )



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> Ergodic Theory

## Modern definition

In modern terminology:

*Ergodic theory is the study of long term average behaviour of systems evolving in time.*



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