Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Budapestlaan 6
3584 CD Utrecht

Measure and Integration 2006-Selected Solutions 13+extra exercises

1. (**Exercise 13.4, p.131**) Denote by λ Lebesgue measure on $(0,1)$. Show that the following iterated integrals exist, but yield different values:

$$\int_{(0,1)}\int_{(0,1)}\frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(x)d\lambda(y)\neq\int_{(0,1)}\int_{(0,1)}\frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(y)d\lambda(x).$$

What does this tell about the $(\lambda\times\lambda)$-integral of the function $\frac{x^2-y^2}{(x^2+y^2)^2}$?

Proof: Notice that for each fixed $y\in(0,1)$, the function $x\to\frac{x^2-y^2}{(x^2+y^2)^2}$ is continuous, and is Riemann integrable on $[0,1]$ since

$$\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dx=-\frac{x}{(x^2+y^2)}|_0^1=-\frac{1}{1+y^2}.$$

Furthermore, the function $y\to-\frac{1}{1+y^2}$ is continuous and Riemann integrable on $[0,1]$ since

$$\int_0^1-\frac{1}{1+y^2}dy=-\tan y|_0^1=-\frac{\pi}{4}.$$

Thus,

$$\int_{(0,1)}\int_{(0,1)}\frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(x)d\lambda(y)=\int_0^1\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy=-\frac{\pi}{4}.$$

Similar analysis shows that

$$\int_{(0,1)}\int_{(0,1)}\frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(y)d\lambda(x)=\int_0^1\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx=\frac{\pi}{4}.$$

Thus the two iterated integrals are not equal. This implies that the function $(x,y)\to\frac{x^2-y^2}{(x^2+y^2)^2}$ is not (Lebesgue) $\lambda\times\lambda$ integrable on $(0,1)\times(0,1)$, otherwise the two integrals would be equal. In fact,

$$\begin{aligned}\int_0^1\int_0^1\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|dy\,dx &\geq \int_0^1\int_0^x\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx\\ &= \int_0^1\frac{1}{2x}=\infty.\end{aligned}$$

2. (**Exercise 13.7, p.131**) Consider $([0,1],\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$, λ is Lebesgue measure and μ is counting measure (i.e. $\mu(A)$ = number of elements in A). Let $\Delta=\{x,y)\in[0,1]\times[0,1]:x=y\}$, show that

$$\int_{[0,1]}\int_{[0,1]}1_\Delta(x,y)d\lambda(x)d\mu(y)\neq\int_{[0,1]}\int_{[0,1]}1_\Delta(x,y)d\mu(y)d\lambda(x).$$

Why does not this violate Tonelli's Theorem?

Proof For any $x, y \in [0,1]$, $\Delta_x = \{y \in [0,1] : (x,y) \in \Delta\} = \{x\}$, and $\Delta_y = \{x \in [0,1] : (x,y) \in \Delta\} = \{y\}$. Thus, $\mu(\Delta_x) = \mu(\Delta_y) = 1$ and $\lambda(\Delta_x) = \lambda(\Delta_y) = 0$. Furthermore,

$$1_\Delta(x,y) = 1 \Leftrightarrow 1_{\Delta_x}(y) = 1 \Leftrightarrow 1_{\Delta_y}(x) = 1.$$

Hence,

$$\int_{[0,1]}\int_{[0,1]} 1_\Delta(x,y)d\lambda(x)d\mu(y) = \int_{[0,1]} \lambda(\Delta_y)d\mu(y) = 0,$$

and

$$\int_{[0,1]}\int_{[0,1]} 1_\Delta(x,y)d\mu(y)d\lambda(x) = \int_{[0,1]} \mu(\Delta_x)d\lambda(x) = \lambda([0,1]) = 1.$$

The reason why Tonelli's Theorem does not hold is because the measure μ is **not** σ-finite.

3. Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Let $f : X \to [0,\infty)$, $g : Y \to [0,\infty)$ be $\mathcal{A}/\mathcal{B}(\mathbb{R})$ respectively $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable functions. Define $h : X \times Y \to [0,\infty)$ by $h(x,y) = f(x)g(y)$.

   (i) Show that h is $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.

   (ii) Prove that $\int_{X\times Y} h(x,y)\, d(\mu \times \nu)(x,y) = \int_X f(x)\, d\mu(x) \cdot \int_Y g(y)\, d\nu(y)$.

   Proof (i) Define $\overline{f} : X \times Y \to [0,\infty)$ by $\overline{f}(x,y) = f(x)$ and $\overline{g} : X \times Y \to [0,\infty)$ by $\overline{g}(x,y) = g(y)$. Then $\overline{f}$ and $\overline{g}$ are $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable since for any $B \in \mathcal{B}(\mathbb{R})$, we have $\overline{f}^{-1}(B) = f^{-1}(B) \times Y \in \mathcal{A} \otimes \mathcal{B}$ and $\overline{g}^{-1}(B) = X \times g^{-1}(B) \in \mathcal{A} \otimes \mathcal{B}$. Now, $h(x,y) = f(x)g(y) = \overline{f}(x,y)\overline{g}(x,y)$ is the product of two $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable functions, hence h is $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.

   Proof (ii) Since $h \geq 0$ is measurable, then by Tonelli's Theorem

$$\int_{X\times Y} h(x,y)\, d(\mu\times\nu)(x,y) = \int_X\int_Y f(x)g(y)\, d\nu(y)\, d\mu(x) = \int_X f(x)\, d\mu(x)\cdot\int_Y g(y)\, d\nu(y)$$

4. Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x,y) = e^{-xt}$) that $\int_{[0,\infty)}(e^{-at} - e^{-bt})\frac{1}{t}d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure.

   Proof: Let $f : [a,b] \times [0,\infty)$ be given by $f(x,y) = e^{-xt}$. Then f is continuous (hence measurable) and $f > 0$. By Toneli's theorem

$$\int_{[0,\infty)}\int_{[a,b]} e^{-xt}d\lambda(x)\, d\lambda(t) = \int_{[a,b]}\int_{[0,\infty)} e^{-xt}d\lambda(t)\, d\lambda(x).$$

   For each $t \in [0,\infty)$, the function $x \to e^{-xt}$ is Riemann integrable on $[a,b]$, hence by Theorem 11.8(i),

$$\int_{[a,b]} e^{-xt}d\lambda(x) = \int_a^b e^{-xt}dx = (e^{-at} - e^{-bt})\frac{1}{t}.$$

Thus,

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t).$$

On the other hand, for each $x \in [a,b]$,

$$\lim_{c\to\infty} \int_0^c e^{-xt}\, dt = \frac{1}{x},$$

hence by Corollary 11.9, $\int_{[0,\infty)} e^{-xt} d\lambda(t) = \frac{1}{x}$. Furthermore, the function $\frac{1}{x}$ is Riemann integrable on $[a,b]$, hence

$$\int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x) = \int_a^b \frac{1}{x}\, dx = \log(b/a).$$

Therefore, $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$.

5. (**Exercise 13.9, p.131**) Let $u : \mathbb{R} \to [0,\infty)$ be a Borel measurable function (there is a misprint in the book in the definition of u). Denote by $S[u] = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le u(x)\}$ and $\Gamma[u] = \{(x, u(x)) : x \in \mathbb{R}\}$.

   (i) Show that $S[u] \in \mathcal{B}(\mathbb{R}^2)$.

   (ii) Is $\lambda^2(S[u]) = \int u\, d\lambda$?

   (iii) Show that $\Gamma[u] \in \mathcal{B}(\mathbb{R}^2)$ and that $\lambda^2(\Gamma[u]) = 0$.

   Proof(i): Define $W : \mathbb{R}^2 \to \mathbb{R}^2$ by $W(x,y) = (u(x), y)$. By Theorem 13.10(ii), W is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)$ measurable (or simply notice that $W^{-1}([a,b) \times [c,d)) = u^{-1}([a,b)) \times [c,d) \in \mathcal{B}(\mathbb{R}^2)$). Let $U : \mathbb{R}^2 \to \mathbb{R}$ be given by $U(x,y) = x - y$, then U is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable, and hence the composition $U \circ W(x,y) = U(u(x), y) = u(x) - y$ is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Finally,

$$S[u] = (\mathbb{R} \times [0,\infty)) \cap (U \circ W)^{-1}[0,\infty) \in \mathcal{B}(\mathbb{R}^2).$$

   Proof(ii): The answer is yes. To see that, notice that for each fixed $x \in \mathbb{R}$, one has

$$\mathbf{1}_{S[u]}(x,y) = 1 \;\Leftrightarrow y \in [0, u(x)] \;\Leftrightarrow \mathbf{1}_{[0,u(x)]}(y) = 1.$$

   Thus, by Tonelli's Theorem (or Theorem 13.5), we have

$$\begin{aligned}
\lambda^2(S[u]) &= \textstyle\int_{\mathbb{R}^2} \mathbf{1}_{S[u]}(x,y)\, d\lambda^2(x,y) \\
&= \textstyle\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{S[u]}(x,y)\, d\lambda(y)\, d\lambda(x) \\
&= \textstyle\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{[0,u(x)]}(y)\, d\lambda(y)\, d\lambda(x) \\
&= \textstyle\int_{\mathbb{R}^2} \lambda([0, u(x)])\, d\lambda(x) \\
&= \textstyle\int_{\mathbb{R}^2} u(x)\, d\lambda(x).
\end{aligned}$$

Proof(iii): We use the same notation as in part (i).

$$\Gamma[u] = (U \circ W)^{-1}(\{0\}) \in \mathcal{B}(\mathbb{R}^2).$$

Notice that for each fixed x,

$$\mathbf{1}_{\Gamma[u]}(x,y) = 1 \ \Leftrightarrow \ y = u(x) \ \Leftrightarrow \mathbf{1}_{\{u(x)\}}(y) = 1.$$

Thus,

$$\begin{aligned} \lambda^2(S[u]) &= \int_{\mathbb{R}^2} \mathbf{1}_{\Gamma[u]}(x,y)\, d\lambda^2(x,y) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{\Gamma[u]}(x,y)\, d\lambda(y)\, d\lambda(x) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{\{u(x)\}}(y)\, d\lambda(y)\, d\lambda(x) \\ &= \int_{\mathbb{R}^2} \lambda(\{u(x)\})\, d\lambda(x) = 0. \end{aligned}$$