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Measure and Integration 2006-Selected Solutions 13

1. (**Exercise 13.6, p.131**)

   (i) Prove that $\int_{(0,\infty)} e^{-tx} d\lambda(t) = \frac{1}{x}$ for all $x > 0$.

   (ii) Use (i) and Fubini's Theorem to show that

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\sin x}{x} d\lambda(x) = \frac{\pi}{2}.$$

Proof(i): Since $e^{-tx} > 0$ for all t and x, hence, for each fixed $x > 0$ the sequence $e^{-tx}\mathbf{1}_{(0,n)}(t) \nearrow e^{-tx}\mathbf{1}_{(0,\infty)}$. Furthermore, for each n, the function $t \to e^{-tx}$ is Riemann integrable on $[0, n]$. Thus, by Beppo-Levi, and Theorem 11.8(i),

$$\begin{aligned}
\int_{(0,\infty)} e^{-tx} d\lambda(t) &= \lim_{n\to\infty} \int_{(0,n)} e^{-tx} d\lambda(t) \\
&= \lim_{n\to\infty} \int_{[0,n]} e^{-tx} d\lambda(t) \\
&= \lim_{n\to\infty} \int_0^n e^{-tx} dt \\
&= \lim_{n\to\infty} \frac{-1}{x} e^{-tx}\Big|_0^n = \frac{1}{x}.
\end{aligned}$$

Proof(ii): Note first that the function $\frac{\sin x}{x}$ is **not** Lebesgue integrable on $(0,\infty)$ (see Remark 11.11 on p.97), so we have to be careful in the application of Fubini's Theorem.

Let $I = \lim_{n\to\infty} \int_{(0,n)} \frac{\sin x}{x} d\lambda(x)$, then by part (i),

$$I = \lim_{n\to\infty} \int_{(0,n)} \lim_{k\to\infty} \int_{(0,k)} e^{-tx} \sin x d\lambda(t)\, d\lambda(x).$$

Since $|\int_{(0,k)} e^{-tx} \sin x\, d\lambda(t)| \leq |\frac{\sin x}{x}|$ which is Riemann and Lebesgue integrable on $[0, n]$, hence by Lebesgue Dominated Convergence Therem,

$$I = \lim_{n\to\infty} \lim_{k\to\infty} \int_{(0,n)} \int_{(0,k)} e^{-tx} \sin x\, d\lambda(t)\, d\lambda(x).$$

By Fubini's Theorem, and integration by parts (after replacing the Lebesgue integral by the Riemann integral), we get

$$\begin{aligned}
I &= \lim_{n\to\infty} \lim_{k\to\infty} \int_{(0,k)} \int_{(0,n)} e^{-tx} \sin x\, d\lambda(x)\, d\lambda(t) \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \int_{(0,k)} \frac{1}{t^2+1} \left(1 - e^{-nt}(\cos n + t \sin n)\right) d\lambda(t).
\end{aligned}$$

Now,

$$|\mathbf{1}_{(0,k)}\frac{1}{t^2+1}\left(1-e^{-nt}(\cos n + t\sin n)\right)| \leq \frac{2}{t^2+1}+\frac{te^{-nt}}{t^2+1} \in \mathcal{L}^1((0,\infty)),$$

Thus by Lebesgue Dominated Convergence Therem,

$$I = \lim_{n\to\infty}\int_{(0,\infty)}\frac{1}{t^2+1}\left(1-e^{-nt}(\cos n + t\sin n)\right)d\lambda(t).$$

Finally, on $(0,\infty)$,

$$|\frac{1}{t^2+1}\left(1-e^{-nt}(\cos n + t\sin n)|\right) \leq \frac{2}{t^2+1}+\frac{te^{-nt}}{t^2+1},$$

hence again by Lebesgue Dominated Convergence Therem, we get

$$\begin{aligned} I &= \int_{(0,\infty)}\lim_{n\to\infty}\tfrac{1}{t^2+1}\left(1-e^{-nt}(\cos n + t\sin n)\right)d\lambda(t) \\ &= \int_{(0,\infty)}\tfrac{1}{t^2+1}\,d\lambda(t) \\ &= \arctan t|_0^\infty = \tfrac{\pi}{2}. \end{aligned}$$

2. (**Exercise 13.12, p.132**) Let μ be a bounded measure on $([0,\infty),\mathcal{B}[0,\infty))$.
   (i) Show that $A \in \mathcal{B}[0,\infty)\otimes\mathcal{P}(\mathbb{N})$ **if and only if** $A = \cup_{j\in\mathbb{N}}B_j\times\{j\}$, where $B_j \in \mathcal{B}[0,\infty)$.
   (ii) Show that there exists a unique measure π on $\mathcal{B}[0,\infty)\otimes\mathcal{P}(\mathbb{N})$ satisfying

$$\pi(B\times\{n\}) = \int_B e^{-t}\frac{t^n}{n!}\,d\mu(t).$$

Proof(i): Clearly any set of the form $A = \cup_{j\in\mathbb{N}}B_j\times\{j\}$, where $B_j \in \mathcal{B}[0,\infty)$ belongs to $\mathcal{B}[0,\infty)\otimes\mathcal{P}(\mathbb{N})$. Now suppose $A \in \mathcal{B}[0,\infty)\otimes\mathcal{P}(\mathbb{N})$. For each $k\in\mathbb{N}$, let $A_k = \{x\in[0,\infty):(x,k)\in A\}$. Notice that $\mathbf{1}_A(x,k) = \mathbf{1}_{A_k}(x)$. By Theorem 13.5, for any $k\in\mathbb{N}$, the function $x\to\mathbf{1}_A(x,k)=\mathbf{1}_{A_k}(x)$ is $\mathcal{B}[0,\infty)$-measurable, hence $A_k\in\mathcal{B}[0,\infty)$. Finally, notice that $A = \cup_{k\in\mathbb{N}}A_k\times\{k\}$.

Proof(ii): Let ν be counting measure on $(\mathbb{N},\mathcal{P}(\mathbb{N}))$. From example 9.10(ii), for any non-negative (measurable) function f on $\mathbb{N}$, and for any $M\subset\mathbb{N}$ one has,

$$\int_M f\,d\nu = \sum_{n\in M} f(n).$$

Consider the product measure $\mu\times\nu$ (note that the underlying measure spaces are σ-finite). The function $f:[0,\infty)\times\mathbb{N}\to[0,\infty)$ given by $f(t,n) = e^{-t}\frac{t^n}{n!}$ is non-negative and measurable (can you see why?). Furthermore, the set function $\pi:\mathcal{B}[0,\infty)\otimes\mathcal{P}(\mathbb{N})\to[0,\infty]$ given by

$$\pi(C) = \int_C e^{-t}\frac{t^n}{n!}\,d(\mu\times\nu)(t,n) = \int_{[0,\infty)\times\mathbb{N}}\mathbf{1}_C e^{-t}\frac{t^n}{n!}\,d(\mu\times\nu)(t,n),$$

defines a measure on $\mathcal{B}[0,\infty)\otimes\mathcal{P}(\mathbb{N})$ (exercise 9.5, p.74), and clearly

$$\pi(B\times\{n\}) = \int_B e^{-t}\frac{t^n}{n!}\,d\mu(t).$$

It remains to show that π is unique. Let $B\in\mathcal{B}[0,\infty)$ and $M\in\mathcal{P}(\mathbb{N})$, by Tonelli's Theorem,

$$\begin{aligned}\pi(B\times M) &= \textstyle\int_B\int_M e^{-t}\frac{t^n}{n!}\,d\nu(n)\,d\mu(t)\\ &= \textstyle\int_B\sum_{n\in M} e^{-t}\frac{t^n}{n!}\,d\mu(t)\\ &= \textstyle\sum_{n\in M}\int_B e^{-t}\frac{t^n}{n!}\,d\mu(t) < \infty.\end{aligned}$$

The last inequality follows from the fact that $|e^{-t}\frac{t^n}{n!}|\le 1$ and μ is a bounded measure. The uniqueness of π follows from a simple application of Theorem 5.7 (note that $[0,\infty)\times\{1,2,\ldots,k\}\nearrow[0,\infty)\times\mathbb{N}$ is an exhaustung sequence of finite π measure).