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Measure and Integration 2006-Selected Solutions 13

1. (**Exercise 13.6, p.131**)

   (i) Prove that $\int_{(0,\infty)} e^{-tx} d\lambda(t) = \frac{1}{x}$ for all $x > 0$.

   (ii) Use (i) and Fubini's Theorem to show that

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\sin x}{x} d\lambda(x) = \frac{\pi}{2}.$$

Proof(i): Since $e^{-tx} > 0$ for all t and x, hence, for each fixed $x > 0$ the sequence $e^{-tx}\mathbf{1}_{(0,n)}(t) \nearrow e^{-tx}\mathbf{1}_{(0,\infty)}$. Furthermore, for each n, the function $t \to e^{-tx}$ is Riemann integrable on $[0, n]$. Thus, by Beppo-Levi, and Theorem 11.8(i),

$$\begin{aligned}
\int_{(0,\infty)} e^{-tx} d\lambda(t) &= \lim_{n\to\infty} \int_{(0,n)} e^{-tx} d\lambda(t) \\
&= \lim_{n\to\infty} \int_{[0,n]} e^{-tx} d\lambda(t) \\
&= \lim_{n\to\infty} \int_0^n e^{-tx} dt \\
&= \lim_{n\to\infty} \frac{-1}{x} e^{-tx}|_0^n = \frac{1}{x}.
\end{aligned}$$

Proof(ii): Note first that the function $\frac{\sin x}{x}$ is **not** Lebesgue integrable on $(0, \infty)$ (see Remark 11.11 on p.97), so we have to be careful in the application of Fubini's Theorem.

Let $I = \lim_{n\to\infty} \int_{(0,n)} \frac{\sin x}{x} d\lambda(x)$, then by part (i),

$$I = \lim_{n\to\infty} \int_{(0,n)} \lim_{k\to\infty} \int_{(0,k)} e^{-tx} \sin x d\lambda(t)\, d\lambda(x).$$

Since $|\int_{(0,k)} e^{-tx} \sin x\, d\lambda(t)| \le |\frac{\sin x}{x}|$ which is Riemann and Lebesgue integrable on $[0, n]$, hence by Lebesgue Dominated Convergence Therem,

$$I = \lim_{n\to\infty} \lim_{k\to\infty} \int_{(0,n)} \int_{(0,k)} e^{-tx} \sin x\, d\lambda(t)\, d\lambda(x).$$

By Fubini's Theorem, and integration by parts (after replacing the Lebesgue integral by the Riemann integral), we get

$$\begin{aligned}
I &= \lim_{n\to\infty} \lim_{k\to\infty} \int_{(0,k)} \int_{(0,n)} e^{-tx} \sin x\, d\lambda(x)\, d\lambda(t) \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \int_{(0,k)} \frac{1}{t^2+1} \left(1 - e^{-nt}(\cos n + t \sin n)\right) d\lambda(t).
\end{aligned}$$

Now,

$$|\mathbf{1}_{(0,k)}\frac{1}{t^2+1}\left(1-e^{-nt}(\cos n+t\sin n)\right)| \le \frac{2}{t^2+1}+\frac{te^{-nt}}{t^2+1}\in\mathcal{L}^1((0,\infty)),$$

Thus by Lebesgue Dominated Convergence Therem,

$$I=\lim_{n\to\infty}\int_{(0,\infty)}\frac{1}{t^2+1}\left(1-e^{-nt}(\cos n+t\sin n)\right)d\lambda(t).$$

Finally, on $(0,\infty)$,

$$|\frac{1}{t^2+1}\left(1-e^{-nt}(\cos n+t\sin n)\right)| \le \frac{2}{t^2+1}+\frac{te^{-t}}{t^2+1},$$

hence again by Lebesgue Dominated Convergence Therem, we get

$$\begin{aligned} I &= \int_{(0,\infty)} \lim_{n\to\infty} \tfrac{1}{t^2+1}\left(1-e^{-nt}(\cos n+t\sin n)\right)d\lambda(t) \\ &= \int_{(0,\infty)} \tfrac{1}{t^2+1}\,d\lambda(t) \\ &= \arctan t|_0^\infty = \tfrac{\pi}{2}. \end{aligned}$$

2. (**Exercise 13.9, p.131**) Let $u:\mathbb{R}\to[0,\infty)$ be a Borel measurable function (there is a misprint in the book in the definition of u). Denote by $S[u]=\{(x,y)\in\mathbb{R}^2: 0\le y\le u(x)\}$ and $\Gamma[u]=\{(x,u(x)):x\in\mathbb{R}\}$.

   (i) Show that $S[u]\in\mathcal{B}(\mathbb{R}^2)$.

   (ii) Is $\lambda^2(S[u])=\int u\,d\lambda$?

   (iii) Show that $\Gamma[u]\in\mathcal{B}(\mathbb{R}^2)$ and that $\lambda^2(\Gamma[u])=0$.

   Proof(i): Define $W:\mathbb{R}^2\to\mathbb{R}^2$ by $W(x,y)=(u(x),y)$. By Theorem 13.10(ii), W is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)$ measurable (or simply notice that $W^{-1}([a,b)\times[c,d))=u^{-1}([a,b))\times[c,d)\in\mathcal{B}(\mathbb{R}^2)$). Let $U:\mathbb{R}^2\to\mathbb{R}$ be given by $U(x,y)=x-y$, then U is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable, and hence the composition $U\circ W(x,y)=U(u(x),y)=u(x)-y$ is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Finally,

   $$S[u]=(\mathbb{R}\times[0,\infty))\cap(U\circ W)^{-1}[0,\infty)\in\mathcal{B}(\mathbb{R}^2).$$

   Proof(ii): The answer is yes. To see that, notice that for each fixed $x\in\mathbb{R}$, one has

   $$\mathbf{1}_{S[u]}(x,y)=1 \;\Leftrightarrow y\in[0,u(x)] \;\Leftrightarrow \mathbf{1}_{[0,u(x)]}(y)=1.$$

   Thus, by Tonelli's Theorem (or Theorem 13.5), we have

   $$\begin{aligned} \lambda^2(S[u]) &= \int_{\mathbb{R}^2}\mathbf{1}_{S[u]}(x,y)\,d\lambda^2(x,y) \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbf{1}_{S[u]}(x,y)\,d\lambda(y)\,d\lambda(x) \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbf{1}_{[0,u(x)]}(y)\,d\lambda(y)\,d\lambda(x) \\ &= \int_{\mathbb{R}^2}\lambda([0,u(x)])\,d\lambda(x) \\ &= \int_{\mathbb{R}^2}u(x)\,d\lambda(x). \end{aligned}$$

Proof(iii): We use the same notation as in part (i).

$$\Gamma[u] = (U \circ W)^{-1}(\{0\}) \in \mathcal{B}(\mathbb{R}^2).$$

Notice that for each fixed x,

$$\mathbf{1}_{\Gamma[u]}(x,y) = 1 \;\Leftrightarrow\; y = u(x) \;\Leftrightarrow \mathbf{1}_{\{u(x)\}}(y) = 1.$$

Thus,

$$\begin{aligned} \lambda^2(S[u]) &= \int_{\mathbb{R}^2} \mathbf{1}_{\Gamma[u]}(x,y)\, d\lambda^2(x,y) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{\Gamma[u]}(x,y)\, d\lambda(y)\, d\lambda(x) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{\{u(x)\}}(y)\, d\lambda(y)\, d\lambda(x) \\ &= \int_{\mathbb{R}^2} \lambda(\{u(x)\})\, d\lambda(x) = 0. \end{aligned}$$

3. (**Exercise 13.11, p.131**) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces such that $\mathcal{A} \neq \mathcal{P}(X)$, and such that $\mathcal{B}$ contains non-empty null-sets.

   (i) Show that the product space $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is never complete even if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are both complete.

   (ii) Conclude that neither $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), \lambda \times \lambda)$ nor the product of the completed spaces $(\mathbb{R}^2, \mathcal{B}^*(\mathbb{R}) \otimes \mathcal{B}^*(\mathbb{R}), \overline{\lambda} \times \overline{\lambda})$ are complete.

   Proof(i): The proof is done by contradiction. Assume that $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is complete. Let $Z \in \mathcal{P}(X) \setminus \mathcal{A}$ en $N \in \mathcal{B}$ a ν-null set. By exercise 13.2 (p. 130), the set $X \times N$ is a $\mu \times \nu$ null-set. Since $Z \times N \subset X \times N$ and $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is complete, it follows that $Z \times N$ is also a $\mu \times \nu$ null-set and hence is $\mathcal{A} \otimes \mathcal{B}$ measurable. By Theorem 13.5, the mapping $x \longrightarrow \mathbf{1}_{Z \times N}(x,y) = \mathbf{1}_Z(x)\mathbf{1}_N(y)$ is $\mathcal{A}$-measurable (note that $\mathbf{1}_N(y)$ is a constant). This implies that $\mathbf{1}_Z$ is $\mathcal{A}$-measurable, and hence $Z \in \mathcal{A}$, which is a contradiction. Hence, $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is not complete.

   Proof(ii): This a direct consequence of part (i).

4. (**Exercise 13.12, p.132**) Let μ be a bounded measure on $([0,\infty), \mathcal{B}[0,\infty))$.

   (i) Show that $A \in \mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$ **if and only if** $A = \cup_{j \in \mathbb{N}} B_j \times \{j\}$, where $B_j \in \mathcal{B}[0,\infty)$.

   (ii) Show that there exists a unique measure π on $\mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$ satisfying

   $$\pi(B \times \{n\}) = \int_B e^{-t} \frac{t^n}{n!}\, d\mu(t).$$

   Proof(i): Clearly any set of the form $A = \cup_{j \in \mathbb{N}} B_j \times \{j\}$, where $B_j \in \mathcal{B}[0,\infty)$ belongs to $\mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$. Now suppose $A \in \mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$. For each $k \in \mathbb{N}$, let $A_k = \{x \in [0,\infty) : (x,k) \in A\}$. Notice that $\mathbf{1}_A(x,k) = \mathbf{1}_{A_k}(x)$. By Theorem 13.5, for any $k \in \mathbb{N}$, the function $x \to \mathbf{1}_A(x,k) = \mathbf{1}_{A_k}(x)$ is $\mathcal{B}[0,\infty)$-measurable, hence $A_k \in \mathcal{B}[0,\infty)$. Finally, notice that $A = \cup_{k \in \mathbb{N}} A_k \times \{k\}$.

Proof(ii): Let ν be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. From example 9.10(ii), for any non-negative (measurable) function f on $\mathbb{N}$, and for any $M \subset \mathbb{N}$ one has,

$$\int_M f\, d\nu = \sum_{n \in M} f(n).$$

Consider the product measure $\mu \times \nu$ (note that the underlying measure spaces are σ-finite). The function $f : [0,\infty) \times \mathbb{N} \to [0,\infty)$ given by $f(t,n) = e^{-t}\frac{t^n}{n!}$ is non-negative and measurable (can you see why?). Furthermore, the set function $\pi : \mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N}) \to [0,\infty]$ given by

$$\pi(C) = \int_C e^{-t}\frac{t^n}{n!}\, d(\mu \times \nu)(t,n) = \int_{[0,\infty)\times\mathbb{N}} \mathbf{1}_C e^{-t}\frac{t^n}{n!}\, d(\mu \times \nu)(t,n),$$

defines a measure on $\mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$ (exercise 9.5, p.74), and clearly

$$\pi(B \times \{n\}) = \int_B e^{-t}\frac{t^n}{n!}\, d\mu(t).$$

It remains to show that π is unique. Let $B \in \mathcal{B}[0,\infty)$ and $M \in \mathcal{P}(\mathbb{N})$, by Tonelli's Theorem,

$$\begin{aligned} \pi(B \times M) &= \int_B \int_M e^{-t}\tfrac{t^n}{n!}\, d\nu(n)\, d\mu(t) \\ &= \int_B \sum_{n\in M} e^{-t}\tfrac{t^n}{n!}\, d\mu(t) \\ &= \sum_{n\in M} \int_B e^{-t}\tfrac{t^n}{n!}\, d\mu(t) < \infty. \end{aligned}$$

The last inequality follows from the fact that $|e^{-t}\frac{t^n}{n!}| \le 1$ and μ is a bounded measure. The uniqueness of π follows from a simple application of Theorem 5.7 (note that $[0,\infty) \times \{1,2,\ldots,k\} \nearrow [0,\infty) \times \mathbb{N}$ is an exhaustung sequence of finite π measure).