



Measure and Integration 2007-Selected Solutions Chapter 7

1. (**Exercise 7.7, p.54**). Use image measures to give a new proof that  $\lambda^n(t \cdot B) = t^n \lambda^n(B)$  for all  $B \in \mathcal{B}(\mathbb{R}^n)$  and for all  $t > 0$ .

**Proof:** Let  $t > 0$ , define  $T_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T_t(x) = \frac{1}{t}x$ , i.e.  $T_t(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$ . Clearly  $T_t$  is continuous and hence measurable. Notice that  $T_t^{-1}B = t \cdot B$ , hence  $T_t(\lambda^n)(B) = \lambda^n(T_t^{-1}(B)) = \lambda^n(t \cdot B)$  for all  $B \in \mathcal{B}(\mathbb{R}^n)$ . Since  $T_t^{-1}$  is a linear transformation, we have

$$T_t(\lambda^n)(B + x) = \lambda^n(T_t^{-1}(B) + T^{-1}x) = \lambda^n(T_t^{-1}(B)) = T_t(\lambda^n)(B).$$

Thus, the measure  $T_t(\lambda^n)$  is translation invariant. Further, if  $I = \prod_{i=1}^n [0, 1)$ , then

$$T_f(\lambda^n)(I) = \lambda^n(t \cdot \prod_{i=1}^n [0, 1)) = \lambda^n(\prod_{i=1}^n [0, t)) = t^n.$$

By Theorem 5.8(ii), we have that  $T_f(\lambda^n) = t^n \lambda^n$ . Hence,  $T_f(\lambda^n)(B) = \lambda^n(t \cdot B) = t^n \lambda^n(B)$  for all  $B \in \mathcal{B}(\mathbb{R}^n)$  and for all  $t > 0$ .

2. (**Exercise 7.8, p.54**) Let  $T : X \rightarrow Y$  be any map. Show that  $T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$  holds for arbitrary families  $\mathcal{G}$  of subsets of  $Y$ .

**Proof:** First note that  $T^{-1}(\sigma(\mathcal{G}))$  is a  $\sigma$ -algebra (see example 3.3(vii)). Since  $\mathcal{G} \subset \sigma(\mathcal{G})$ , then  $T^{-1}\mathcal{G} \subset T^{-1}(\sigma(\mathcal{G}))$  and hence  $\sigma(T^{-1}(\mathcal{G})) \subset T^{-1}(\sigma(\mathcal{G}))$  (since  $\sigma(T^{-1}(\mathcal{G}))$  is the smallest  $\sigma$ -algebra containing  $T^{-1}(\mathcal{G})$ ). Furthermore,  $T^{-1}(\mathcal{G}) \subset \sigma(T^{-1}(\mathcal{G}))$ , hence by Lemma 7.2, we have  $T^{-1}(\sigma(\mathcal{G})) \subset \sigma(T^{-1}(\mathcal{G}))$ . Thus,  $T^{-1}(\sigma(\mathcal{G})) = \sigma(T^{-1}(\mathcal{G}))$ .

3. (**Exercise 7.10, p.55**) (Cantor's ternary set). Let  $(X, \mathcal{A}) = [0, 1], [0, 1] \cap \mathcal{B}(\mathbb{R})$ ,  $\lambda = \lambda^1|_{[0,1]}$ , and  $E_0 = [0, 1]$ . Remove the open middle third of  $E_0$  to get two disjoint intervals whose union is  $E_1$ . Remove the open middle third from each of the intervals making up  $E_1$  to get four disjoint intervals whose union is  $E_2$ , etc...

(i) Describe explicitly  $E_0, E_1, E_2, E_3$ .

**Proof:**  $E_0 = [0, 1]$ ,  $E_1 = [0, 1/3] \cup [2/3, 1]$ ,  
 $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$   
 $E_3 = [0, 1/27] \cup [2/27, 1/9] \cup [2/9, 7/27] \cup [8/27, 1/3] \cup [2/3, 19/27] \cup [20/27, 7/9] \cup [8/9, 25/27] \cup [26/27, 1]$ .

(ii) Prove that each  $E_n$  is compact (i.e. closed and bounded). Conclude that  $C = \cup_{n \geq 0} E_n$  is non-empty and compact.

**Proof:** Each  $E_n$  is a finite union of  $2^n$  compact intervals, hence  $E_n$  is compact (note that finite union of closed and bounded sets is itself closed and bounded). Since  $E_0 \supset E_1 \supset E_2 \supset \dots$ , then  $C$  is also compact (note that arbitrary intersection of closed sets is closed and since each set is bounded then  $C$  is closed and bounded). We now prove that  $C$  is non-empty. For each  $n$ , choose a point  $x_n \in E_n$  (note that  $x_n \in E_i$  for  $i \leq n$ ). Then  $(x_n)$  is a sequence in  $E_0 = [0, 1]$ . Since  $E_0$  is compact, then  $(x_n)$  has a convergent subsequence  $(x_{n_i})$  converging to some point  $x \in E_0$ . Notice that for each  $n$   $x_{n_i} \in E_n$  for all  $n_i$  sufficiently large (namely for all  $n_i \geq n$ ). Since  $E_n$  is compact, then the limit point  $x$  must lie in  $E_n$  for all  $n$ . Hence,  $x \in C$  and  $C$  is non-empty.

(iii) Prove that  $C \cap \bigcup_{n \in \mathbb{N}} \bigcup_{k \geq 0} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) = \emptyset$ .

**Proof:** Notice that at stage  $n$ , the sets removed from  $E_{n-1}$  in order to get  $E_n$  are all of the form  $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$  with  $n > 0$  and  $k \geq 0$ . Hence,  $C$  is disjoint from any such interval and therefore disjoint from the union of such intervals.

(iv) Determine  $\lambda(E_n)$  and show that  $\lambda(C) = 0$ .

**Proof:** For each  $n$ ,  $E_n$  consists of  $2^n$  disjoint closed interval each of length (or Lebesgue measure) equals  $3^{-n}$ . Hence,  $\lambda(E_n) = 2^n 3^{-n} = \left(\frac{2}{3}\right)^n$ . Since  $\lambda$  on  $[0, 1]$  is a finite measure, and  $E_n$  is a sequence decreasing to  $C$ , we have by Theorem 4.4 that  $\lambda(C) = \lim_{n \rightarrow \infty} \lambda(E_n) = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0$ .

(v) Show that  $C$  does not contain any non-empty open interval, and that the interior of  $C$  is empty.

**Proof:** The proof is by contradiction. Suppose that  $C$  contains a non-empty open interval  $(a, b)$ . Let  $\epsilon = b - a$ . Notice that  $\epsilon > 0$  since  $(a, b)$  is non-empty. Choose  $N$  sufficiently large so that  $3^{-N} < \epsilon$  for all  $n \geq N$ . Since  $E_n$  consists of  $2^n$  disjoint intervals each of length  $3^{-n}$ , then  $(a, b)$  cannot be contained in any of these intervals if  $n \geq N$  ( $(a, b)$  has length bigger than  $3^{-n}$ ), and hence  $(a, b)$  cannot be contained in  $E_n$  for all  $n \geq N$ . Therefore  $(a, b)$  cannot be contained in  $C$ , a contradiction. Finally, if  $x \in C$  is an interior point, then  $C$  must contain an open interval containing  $x$ , but from the above, this is impossible. Hence  $C$  contains no interior points, in other words, the interior of  $C$  is empty.

(vi) Show that every point  $x \in [0, 1]$  has a ternary expansion of the form  $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$  with  $a_i \in \{0, 1, 2\}$ , which we also denote by  $x = 0.a_1 a_2 \dots$ . Conclude that  $C$  consists of points whose ternary expansion does not contain the digit 1, i.e.,  $a_i \in \{0, 2\}$ .

**Proof:** We first show how the ternary expansion is obtained. To determine the first digit, we divide the unit interval into three intervals of equal length (namely,  $[0, 1/3]$ ,  $(1/3, 2/3)$ ,  $[2/3, 1]$ ). Points in the left interval are assigned  $a_1 = 0$ , points in the middle piece are assigned  $a_1 = 1$  and points in the right piece are assigned  $a_1 = 2$ . To determine the second digit  $a_2$ , we divide each of the intervals in the previous stage into three intervals of equal length (namely  $1/9$ ). The left piece

is assigned  $a_2 = 0$ , the middle piece is assigned  $a_2 = 1$  and the right piece is assigned  $a_2 = 2$ . This process is repeated indefinitely, and at stage  $n$  the digit  $a_n$  is determined according to its location in the left piece ( $a_n = 0$ ), or the middle piece ( $a_n = 1$ ), or the right piece ( $a_n = 2$ ). In this way, we see that each point in  $[0, 1]$  has a ternary expansion. In fact all points have a unique ternary expansion except for points of the form  $k/3^n$ , these have exactly two ternary expansion. For example  $1/3 = 0.100000\dots = 0.022222\dots$ . In the above description, the middle piece is chosen to be open, hence in the above procedure the ternary expansion obtained for  $1/3$  is  $0.022222\dots$

By the construction of the Cantor set, we see that at stage 1, the middle piece is removed, i.e. all points whose ternary expansion starts with 1 are removed, and only points with  $a_1 \in \{0, 2\}$  remain. At stage 2, we remove from the remaining pieces the middle piece of each interval, i.e., we remove all points whose second ternary digit  $a_2 = 1$ . So only points whose ternary expansion starting with  $a_1, a_2 \in \{0, 2\}$  remain. If we continue this process, we see that at stage  $n$ , only points whose first  $n$  digits  $a_1, a_2, \dots, a_n \in \{0, 2\}$  remain. The limiting process, namely  $C$ , consists then of all points whose ternary expansion does not contain the digit 1.

(vii) Show that  $C$  is uncountable (yet  $\lambda(C) = 0$ ).

**Proof:** For this we need to define a bijection from  $C$  onto  $[0, 1]$ . Define  $f : C \rightarrow [0, 1]$  as follows: let  $x \in C$ , then  $x$  has a ternary expansion of the form  $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$  with

$a_i \in \{0, 2\}$ . Set  $f(x) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$  with  $b_i = a_i/2$ . Then, it is easy to check that  $f$  is a bijection. Since  $[0, 1]$  is uncountable, it follows that  $C$  is uncountable.