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Measure and Integration 2006-Selected Solutions Chapter 7/8

1. (**Exercise 7.7, p.54**). Use image measures to give a new proof that $\lambda^n(t\cdot B) = t^n\lambda^n(B)$ for all $B\in\mathcal{B}(\mathbb{R}^n)$ and for all $t>0$.

   Proof: Let $t>0$, define $T_t:\mathbb{R}^n\to\mathbb{R}^n$ by $T_t(x)=\frac{1}{t}x$, i.e. $T_t(x_1,\cdots,x_n) = (tx_1,\cdots,tx_n)$. Clearly T_t is continuous and hence measurable. Notice that $T_t^{-1}B = t\cdot B$, hence $T_t(\lambda^n)(B)=\lambda^n(T_t^{-1}(B))=\lambda^n(t\cdot B)$ for all $B\in\mathcal{B}(\mathbb{R}^n)$. Since T_t^{-1} is a linear transformation, we have

   $$T_t(\lambda^n)(B+x)=\lambda^n(T_t^{-1}(B)+T^{-1}x)=\lambda^n(T_t^{-1}(B))=T_t(\lambda^n)(B).$$

   Thus, the measure $T_t(\lambda^n)$ is translation invariant. Further, if $I=\prod_{i=1}^n[0,1)$, then

   $$T_f(\lambda^n)(I)=\lambda^n(t\cdot\prod_{i=1}^n[0,1))=\lambda^n\prod_{i=1}^n[0,t))=t^n.$$

   By Theorem 5.8(ii), we have that $T_f(\lambda^n)=t^n\lambda^n$. Hence, $T_f(\lambda^n)(B)=\lambda^n(t\cdot B) = t^n\lambda^n(B)$ for all $B\in\mathcal{B}(\mathbb{R}^n)$ and for all $t>0$.

2. (**Exercise 8.2, p.65**) Define

   $$\mathcal{B}(\overline{\mathbb{R}})=\{B\cup C: B\in\mathcal{B}(\mathbb{R}),\ C\in\{\emptyset,\{-\infty\},\{+\infty\},\{-\infty,+\infty\}\}.$$

   Show that $\mathcal{B}(\overline{\mathbb{R}})$ is a σ-algebra over $\overline{\mathbb{R}}$. Moreover prove that $\mathcal{B}(\mathbb{R})=\mathbb{R}\cap\mathcal{B}(\overline{\mathbb{R}})$, and $\mathcal{O}(\mathbb{R})=\mathcal{O}(\overline{\mathbb{R}})\cap\mathbb{R}$ where $\mathcal{O}(\overline{\mathbb{R}})$ is the usual topology on $\overline{\mathbb{R}}$.

   Proof: Clearly $\emptyset\in\mathcal{B}(\overline{\mathbb{R}})$ since $\emptyset=\emptyset\cup\emptyset$ and $\emptyset\in\mathcal{B}(\mathbb{R})$. Suppose $\overline{B}\in\mathcal{B}(\overline{\mathbb{R}})$, then $\overline{B}=B\cup C$ with $B\in\mathcal{B}(\mathbb{R})$ and $C\in\{\emptyset,\{-\infty\},\{+\infty\},\{-\infty,+\infty\}$. Now, $\overline{B}^c=\overline{\mathbb{R}}\setminus\overline{B}=(\overline{\mathbb{R}}\setminus B)\cap(\overline{\mathbb{R}}\setminus C)$. Since $B\subset\mathbb{R}$, then $\overline{\mathbb{R}}\setminus B$ contains $\{-\infty,+\infty\}$, so that $\overline{\mathbb{R}}\setminus B=(\mathbb{R}\setminus B)\cup\{-\infty,+\infty\}$. Furthermore, $\mathbb{R}\subset\overline{\mathbb{R}}\setminus C$, hence

   $$\begin{aligned}\overline{B}^c &= [(\mathbb{R}\setminus B)\cup\{-\infty,+\infty\}]\cap(\overline{\mathbb{R}}\setminus C)\\ &= [(\mathbb{R}\setminus B)\cap(\overline{\mathbb{R}}\setminus C)]\cup[\{-\infty,+\infty\}\cap(\overline{\mathbb{R}}\setminus C)]\\ &= (\mathbb{R}\setminus B)\cup[\{-\infty,+\infty\}\cap(\overline{\mathbb{R}}\setminus C)].\end{aligned}$$

   Since $\mathbb{R}\setminus B\in\mathcal{B}(\mathbb{R})$ and $\{-\infty,+\infty\}\cap(\overline{\mathbb{R}}\setminus C)\in\{\emptyset,\{-\infty\},\{+\infty\},\{-\infty,+\infty\}\}$, it follows that $\overline{B}^c\in\mathcal{B}(\overline{\mathbb{R}})$. Finally, let $\overline{B}_n\in\mathcal{B}(\overline{\mathbb{R}})$. Then, $\overline{B}_n=B_n\cup C_n$ with $B_n\in\mathcal{B}(\mathbb{R})$ and $C_n\in\{\emptyset,\{-\infty\},\{+\infty\},\{-\infty,+\infty\}\}$. Now,

   $$\bigcup_n\overline{B}_n=(\bigcup_n B_n)\cup(\bigcup_n C_n)\in\mathcal{B}(\overline{\mathbb{R}}),$$

since $\bigcup_n B_n \in \mathcal{B}(\overline{\mathbb{R}})$ and $\bigcup_n C_n \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$.

We now show that $\mathcal{B}(\mathbb{R}) = \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$. Clearly, $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$. Now let $D \in \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$, then $D = \overline{B} \cap \mathbb{R}$ where $\overline{B} = B \cup C$ with $B \in \mathcal{B}(\mathbb{R})$ and $C \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}$. Hence, $D = B \in \mathcal{B}(\mathbb{R})$. This shows that $\mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R} \subset \mathcal{B}(\mathbb{R})$, and therefore $\mathcal{B}(\mathbb{R}) = \mathcal{B}(\overline{\mathbb{R}}) \cap \mathbb{R}$.

Finally, we need to prove that $\mathcal{O}(\mathbb{R}) = \mathcal{O}(\overline{\mathbb{R}}) \cap \mathbb{R}$ where $\mathcal{O}(\overline{\mathbb{R}})$ is the usual topology on $\overline{\mathbb{R}}$. To see this note that a set $\overline{U} \in \overline{\mathbb{R}}$ is open if for each point $x \in \overline{U}$ there is an open neighborhood containing x and is contained in $\overline{U}$. If x is real, then an open neighboorhood is an open interval in the usual sense. If $x = -\infty$, then an open neighborhood is an interval of the for $[-\infty, a)$, and if $x = +\infty$, then an open neighborhood is an interval of the form $(a, +\infty]$. In otherwords, a set $\overline{U} \subset \overline{\mathbb{R}}$ is open if it is of the form $\overline{U} = U \cup C$, where U is open in $\mathbb{R}$ and C of the form $[-\infty, a)$ or $(a, +\infty]$ or $\emptyset$ or $\overline{\mathbb{R}}$ or union of those. Therefore, $\mathcal{O}(\mathbb{R}) = \mathbb{R} \cap \mathcal{O}(\overline{\mathbb{R}})$.

3. (**Exercise 8.9, p.65**) Show that the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \max\{x, 0\}$ and $g(x) = \min\{x, 0\}$ are continuous and hence $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. Conclude that if $(X, \mathcal{A})$ is a measure space and $u : X \to \mathbb{R}$ is an $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable function, then the positive part u^+ and the negative part u^- are also $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable.

   Proof: Notice that the functions $i, k : \mathbb{R} \to \mathbb{R}$ given by $i(x) = x$ and $k(x) = |x|$ are continuous. Now, $f(x) = \frac{1}{2}(i(x) + k(x))$ and $g(x) = \frac{1}{2}(-i(x) + k(x))$ are linear combinations of continuous functions, hence continuous and therefore, $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. Finally, if $u : X \to \mathbb{R}$ is an $\mathcal{A}/\mathcal{B}(\mathbb{R})$ measurable function, then $u^+ = f \circ u$ and $u^- = g \circ u$ are compositions of measurable functions, hence measurable.

4. (**Exercise 8.15, p.65**) Let λ be the one-dimensional Lebesgue measure and $u : \mathbb{R} \to \mathbb{R}$ given by $u(x) = |x|$. Determine the measure $u(\lambda) = \lambda \circ u^{-1}$.

   Proof: Notice that $u(\mathbb{R}) = [0, \infty)$. Hence for all Borel sets $B \subset (-\infty, 0)$, one has $u(\lambda)(B) = \lambda(u^{-1}(B)) = \lambda(\emptyset) = 0$. We therefore need to determine $\lambda \circ u^{-1}$ on $\mathcal{B}(\mathbb{R}) \cap [0, \infty)$. Suppose $(a, b) \subset [0, \infty)$ is an interval, then

$$\begin{aligned} u(\lambda)(a, b) &= \lambda(u^{-1}((a, b))) = \lambda((-b, -a) \cup (a, b)) \\ &= (-a - (-b)) + (b - a) = 2(b - a) = 2\lambda((a, b)). \end{aligned}$$

   Since $\{[0, k)\}$ is an exhaustung sequence of finite $u(\lambda)$, by Theorem 5.7, we see that $u(\lambda) = 2\lambda$.