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Measure and Integration 2006-Selected Solutions Chapter 8/9

1. (**p. 65 exercise 8.3**) Let $(X, \mathcal{A})$ be a measurable space.

   (a) Let $f, g : X \to \mathbb{R}$ be measurable fuctions and let $A \in \mathcal{A}$. Show that the function $h : X \to R$ defined by $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \notin A$ is measurable.

   (b) Let $(f_j)_j$ be a sequence of measurable functions and let $(A_j)_j \subset \mathcal{A}$ be such that $X = \bigcup_j A_j$ and $f_j = f_k$ on $A_j \cap A_k$. Define $f : X \to \mathbb{R}$ by $f(x) = f_j$ if $x \in A_j$. Show that f is measurable.

   Proof(a): Notice that since $A, A^c \in \mathcal{A}$, then the indicator functions 1_A and 1_{A^c} are measurable. Furthermore, $h(x) = f(x) \cdot 1_A(x) + g(x) \cdot 1_{A^c}$, hence measurable (we used the fact the sums and products of measurable functions are measurable).

   Proof(b): Notice that the condition $f_j = f_k$ on $A_j \cap A_k$ implies that f is well-defined. Let $B \in \mathcal{B}(\mathbb{R})$, then

   $$f^{-1}(B) = f^{-1}(B) \cap \bigcup_j A_j = \bigcup_j (f^{-1}(B) \cap A_j) = \bigcup_j (f_j^{-1}(B) \cap A_j) \in \mathcal{A}.$$

   Therefore, f is meaurable.

2. (**p.65 exercise 8.12**) Let $u : \mathbb{R} \to \mathbb{R}$ be differentiable. Explain why u and $u' = du/dx$ are measurable.

   Proof: Since u is differentiable, then u is continuous and hence measurable. Define $u_n : \mathbb{R} \to \mathbb{R}$ by $u_n = n\,(u(x + 1/n) - u(x))$. Notice that u_n is a linear combination of measurable functions, hence measurable. Also $u'(x) = \lim_{n\to\infty} u_n(x)$ for all $x \in \mathbb{R}$. Therefore by Corollary 8.9, u' is measurable.

3. (**p.73, exercise 3**) Give an example of a sequence (f_n) such that $f_n : \mathbb{R} \to \mathbb{R}$ is an increasing function, but the sequence (f_n) is not increasing.

   Proof: For $k \geq 1$, let $f_k : \mathbb{R} \to \mathbb{R}$ be given by

   $$f_k(x) = \begin{cases} 0 & -\infty < x < -1/k \\ k^3(x + 1/k), & -1/k \leq x \leq 0 \\ k^2, & x \geq 0 \end{cases}$$

   Then, it is easy to see that if $x < y$, then $f_k(x) \leq f_k(y)$ so that f_k is an increasing function for each $k \geq 1$. However, the sequence (f_k) is not increasing since for any k, and for any $x \in (-1/k, -1/(k+1))$, one has $f_k(x) > 0$, while $f_{k+1}(x) = 0$ so that $f_{k+1}(x) < f_k(x)$.