Universiteit Utrecht Universiteit Utrecht Boedapestlaan 6
Mathematisch Instituut 3584 CD Utrecht

Measure and Integration 2007-Selected Solutions Chapter 9

1. **(p.73, exercise 9.1)** Let $(X, \mathcal{A}, \mu)$ be a measure space and f a non-negative simple function such that $f(x) = \sum_{j=1}^m y_j 1_{A_j}(x)$, where $y_i \geq 0$ and $A_j \in \mathcal{A}$ are not necessarily disjoint. Show that $I_\mu(f) = \sum_{j=1}^m y_j \mu(A_j)$.

   Proof: The function f can be seen as a sum of m simple functions. Hence by Properties 9.3((i) and (ii)), we have

$$I_\mu(f) = I_\mu(\sum_{j=1}^m y_j 1_{A_j}) = \sum_{j=1}^m I_\mu(y_j 1_{A_j}) = \sum_{j=1}^m y_j I_\mu(1_{A_j}) = \sum_{j=1}^m y_j \mu(A_j).$$

2. **(p.73, exercise 9.3)** Give an example of a sequence (f_n) such that $f_n : \mathbb{R} \to \mathbb{R}$ is an increasing function, but the sequence (f_n) is not increasing.

   Proof: For $k \geq 1$, let $f_k : \mathbb{R} \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 0 & -\infty < x < -1/k \\ k^3(x + 1/k), & -1/k \leq x \leq 0 \\ k^2, & x \geq 0 \end{cases}$$

   Then, it is easy to see that if $x < y$, then $f_k(x) \leq f_k(y)$ so that f_k is an increasing function for each $k \geq 1$. However, the sequence (f_k) is not increasing since for any k, and for any $x \in (-1/k, -1/(k+1))$, one has $f_k(x) > 0$, while $f_{k+1}(x) = 0$ so that $f_{k+1}(x) < f_k(x)$.

3. **(p.73, exercise 9.7)** Let $(X, \mathcal{A})$ be a measurable space, and $(\mu_j)_{j\in\mathbb{N}}$ a sequence of measures on $(X, \mathcal{A})$. Let $\mu = \sum_{j\in\mathbb{N}} \mu_j$ (by problem 4.6, μ is a measure). Show that for every $u \in \mathcal{M}^+(\mathcal{A})$, one has

$$\int u \, \mu = \sum_{j\in\mathbb{N}} \int u \, \mu_j.$$

   Proof: Suppose first that $u = 1_A$, where $A \in \mathcal{A}$. Then,

$$\int u \, d\mu = \mu(A) = \sum_{n=1}^\infty \mu_n(A) = \sum_{n=1}^\infty \int u \, d\mu_n.$$

Suppose now that $u = \sum_{k=1}^{m} a_k 1_{A_k}$ is a non-negative simple function in standard form, note that $A_1, \cdots, A_m$ are measurable and disjoint. Then,

$$\int f\, d\mu = \sum_{k=1}^{m} a_k \mu(A_k) = \sum_{k=1}^{m} a_k \sum_{n=1}^{\infty} \mu_n(A_k) = \sum_{n=1}^{\infty} \sum_{k=1}^{m} a_k \mu_n(A_k) = \sum_{n=1}^{\infty} \int f\, d\mu_n.$$

Finally, let $u \geq 0$ be measurable. There exists an increasing sequence of non-negative simple functions f_m converging to u. By Theorem 9.6 (Beppo-Levi), $\int u\, d\mu_j = \lim_{m\to\infty} \int f_m\, d\mu_j$ for all $j \in \mathbb{N}$. Consider the double sequence $a_{m,n} = \sum_{j=1}^{n} \int f_m\, d\mu_j$. It is easy to see that $(a_{m,n})$ is increasing in m and in n, hence by exercise 4.6, $\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}$. Now,

$$\begin{aligned}
\int u\, d\mu &= \lim_{m\to\infty} \int f_m\, d\mu \\
&= \lim_{m\to\infty} \sum_{j=1}^{\infty} \int f_m\, d\mu_j \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{j=1}^{n} \int f_m\, d\mu_j \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \int f_m\, d\mu_j \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \lim_{m\to\infty} \int f_m\, d\mu_j \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \int u\, d\mu_j \\
&= \sum_{j=1}^{\infty} \int u\, d\mu_j.
\end{aligned}$$

4. (**p.73, exercise 9.9**) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $(A_j)_{j\in N}$ be a sequence of measurable sets. Set

$$\liminf_{j\to\infty} A_j = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j, \text{ and } \limsup_{j\to\infty} A_j = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j.$$

   (i) Prove that $\mathbf{1}_{\liminf_{j\to\infty} A_j} = \liminf_{j\to\infty} \mathbf{1}_{A_j}$, and $\mathbf{1}_{\limsup_{j\to\infty} A_j} = \limsup_{j\to\infty} \mathbf{1}_{A_j}$.
   (ii) Prove that $\mu(\liminf_{j\to\infty} A_j) \leq \liminf_{j\to\infty} \mu(A_j)$.
   (iii) Prove that if μ is a finite measure, then $\limsup_{j\to\infty} \mu(A_j) \leq \mu(\limsup_{j\to\infty} A_j)$.
   (iv) Provide an example showing that (iii) fails if μ is not finite.

Proof (i): We first begin by proving two general facts, namely if $E = \bigcap_{j=1}^{\infty} E_j$ and if $F = \bigcup_{j=1}^{\infty} F_j$, then $\mathbf{1_E} = \inf_{n\geq 1} \mathbf{1}_{E_j}$ and $\mathbf{1_F} = \sup_{n\geq 1} \mathbf{1}_{F_j}$. We prove the first, the second is proved in a similar way. We need to investigate when both sides are equal to 1. To this end, consider

$$\begin{aligned} \mathbf{1}_{\mathbf{E}}(x) = 1 &\iff x \in E_j \text{ for all } j \geq 1 \\ &\iff \mathbf{1}_{E_j}(x) = 1 \text{ for all } j \geq 1 \\ &\iff \inf_{j\geq 1} \mathbf{1}_{E_j}(x) = 1. \end{aligned}$$

This proves that $\mathbf{1}_{\mathbf{E}} = \inf\limits_{n\geq 1} \mathbf{1}_{E_j}$, and similarly $\mathbf{1}_{\mathbf{F}} = \sup\limits_{n\geq 1} \mathbf{1}_{F_j}$. Going back to the proof of the exercise...we set $B_n = \bigcap\limits_{j=n}^{\infty} A_j$ and $C_n = \bigcup\limits_{j=n}^{\infty} A_j$. Then, $\liminf\limits_{j\to\infty} A_j = \bigcup\limits_{n=1}^{\infty} B_n$ and $\limsup\limits_{j\to\infty} A_j = \bigcap\limits_{n=1}^{\infty} C_n$. By the above we have

$$\mathbf{1}_{\liminf_{j\to\infty} A_j} = \mathbf{1}_{\bigcup_{n=1}^{\infty} B_n} = \sup_{n\geq 1} \mathbf{1}_{B_n} = \sup_{n\geq 1} \mathbf{1}_{\cap_{j=n}^{\infty} A_j} = \sup_{n\geq 1} \inf_{j\geq n} \mathbf{1}_{A_j} = \liminf_{n\to\infty} \mathbf{1}_{A_j},$$

and

$$\mathbf{1}_{\limsup_{j\to\infty} A_j} = \mathbf{1}_{\bigcap_{n=1}^{\infty} C_n} = \inf_{n\geq 1} \mathbf{1}_{C_n} = \inf_{n\geq 1} \mathbf{1}_{\cup_{j=n}^{\infty} A_j} = \inf_{n\geq 1} \sup_{j\geq n} \mathbf{1}_{A_j} = \limsup_{n\to\infty} \mathbf{1}_{A_j}.$$

Proof (ii): Applying Fatou's Lemma to the sequence $(\mathbf{1}_{A_j})$ and using part (i), we get

$$\mu(\liminf_{j\to\infty} A_j) = \int \mathbf{1}_{\liminf_{j\to\infty} A_j} \, d\mu = \int \liminf_{j\to\infty} \mathbf{1}_{A_j} \, d\mu \leq \liminf_{j\to\infty} \int \mathbf{1}_{A_j} \, d\mu = \liminf_{j\to\infty} \mu(A_j).$$

Proof (iii): Notice that $\mathbf{1}_{A_J} \leq 1$ and $\int 1 \, d\mu = \mu(X) < \infty$. Hence, by exercise 9.8 (reverse of Fatou's Lemma) we have

$$\limsup_{j\to\infty} \mu(A_j) = \limsup_{j\to\infty} \int \mathbf{1}_{A_j} \, d\mu \leq \int \limsup_{j\to\infty} \mathbf{1}_{A_j} \, d\mu = \int \mathbf{1}_{\limsup_{j\to\infty} A_j} \, d\mu = \mu(\limsup_{j\to\infty} A_j).$$

Proof (iv): Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ is Lebesgue measure. Notice that $\lambda(\mathbb{R}) = \infty$. For $j \geq 1$, let $A_j = [j, 2j]$. Then,

$$\limsup_{j\to\infty} \mu(A_j) = \inf_{n\to\infty} \sup_{j\geq n} \mu(A_j) = \inf_{n\to\infty} \sup_{j\geq n} j = \infty.$$

On the other hand,

$$\mu(\limsup_{j\to\infty} A_j) = \mu(\cap_{n=1}^{\infty} \cup_{j=n}^{\infty} [j, 2j]) = \mu(\cap_{n=1}^{\infty} [n, \infty)) = \mu(\emptyset) = 0.$$

Hence, (iii) fails in this case.

5. (**p.73, exercise 9.10**) Let $(X, \mathcal{A}, \mu)$ be a measure space and (A_j) a sequence of disjoint measurable sets such that $X = \bigcup\limits_{j\in\mathbb{N}} A_j$.

(i) Show that for every $u \in \mathcal{M}^+(\mathcal{A})$ (i.e. u is a non-negative measurable functions with values in $[0, \infty]$) one has

$$\int u\, d\mu = \sum_{j=1}^{\infty} \int \mathbf{1}_{A_j} u\, d\mu.$$

(ii) Assume $(X, \mathcal{A}, \mu)$ is σ-finite. Use part (i) to construct a measurable function $w > 0$ such that $\int w\, d\mu < \infty$.

Proof (i): From $X = \bigcup_{j\in\mathbb{N}} A_j$ (disjoint union), it is easy to see that $1 = \mathbf{1}_X = \sum_{j=1}^{\infty} \mathbf{1}_{A_j}$. By Corollary 9.9, for any $u \in \mathcal{M}^+(\mathcal{A})$ one has

$$\int u\, d\mu = \int \sum_{j=1}^{\infty} \mathbf{1}_{A_j} u\, d\mu = \sum_{j=1}^{\infty} \int \mathbf{1}_{A_j} u\, d\mu.$$

Proof (ii): Suppose μ is σ-finite, then there exists an increasing sequence (E_n) of measurable sets such that $X = \bigcup_{n\in\mathbb{N}} E_n$, and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Define $A_1 = E_1$, and for $n \geq 2$, $A_n = E_n \setminus E_{n-1}$. Then the sequence (A_n) is disjoint, $X = \bigcup_{j\in\mathbb{N}} A_j$ and $\mu(A_n) = \mu(E_n) - \mu(E_{n-1}) < \infty$. Define w on X by

$$w(x) = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu(A_n) + 1} \mathbf{1}_{A_n}.$$

Then, clearly, $w(x) > 0$ for all $x \in X$, and by Corollary 9.9,

$$\int w\, d\mu = \int \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu(A_n) + 1} \mathbf{1}_{A_n}\, d\mu = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu(A_n) + 1} \int \mathbf{1}_{A_n}\, d\mu$$

$$= \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu(A_n) + 1} \mu(A_n) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$