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Measure and Integration 2007-Selected Solutions Chapter 11

1. (**p.100, exercise 11.1**) Let $(X, \mathcal{A}, \mu)$ be a measure space, and (u_j) a sequence of measrable real valued functions such that $\lim_{j\to\infty} u_j(x) = u(x)$ for all $x \in X$. Suppose that $|u_j| \leq g$ for some measurable function g such that $g^p \in \mathcal{L}^1_+$, $p > 0$. Show that $\lim_{j\to\infty} \int |u_j - u|^p \, d\mu = 0$.

   Proof: First notice that for any $a, b \in \mathbb{R}$, one has

   $$|a-b|^p \leq (|a| + |b|)^p \leq (2\max(|a|, |b|))^p = 2^p \max(|a|^p, |b|^p) \leq 2^p(|a|^p + |b|^p).$$

   Applying this fact to our sequence, we see that $|u_j(x) - u(x)|^p \leq 2^p g^p(x)$ (note that $|u_j| \leq g$ implies $|u| \leq g$), and g^p is a non-negative integrable function. Furthermore, $\lim_{j\to\infty} |u_j - u|^p = 0$, hence by Lebesgue Dominated Convergence Theorem,

   $$\lim_{j\to\infty} \int |u_j - u|^p \, d\mu = \int \lim_{j\to\infty} |u_j - u|^p \, d\mu = 0.$$

2. (**p.100, exercise 11.3**) Let (f_k), (g_k) and (G_k) be sequences of integrable functions on a measure space $(X, \mathcal{A}, \mu)$. If

   (i) $\lim_{k\to\infty} f_k(x) = f(x)$, $\lim_{k\to\infty} g_k(x) = g(x)$ and $\lim_{k\to\infty} G_k(x) = G(x)$ for all $x \in X$,

   (ii) $g_k(x) \leq f_k(x) \leq G_k(x)$ for all $k \geq 1$ and all $x \in X$,

   (iii) $\lim_{k\to\infty} \int g_k \, d\mu = \int g \, d\mu$, $\lim_{k\to\infty} \int G_k \, d\mu = \int G \, d\mu <$ and both $\int g \, d\mu$ and $\int G \, d\mu$ are finite,

   then, $\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$ and $\int f \, d\mu$ is finite.

   Proof: By assumption $0 \leq f_k - g_k \to f - g$ and $0 \leq G_k - f_k \to G - f$. By Fatou's Lemma we have

   $$\begin{aligned}
   \int (f-g)\, d\mu &= \int \lim_{k\to\infty}(f_k - g_k)\, d\mu \\
   &= \int \liminf_{k\to\infty}(f_k - g_k)\, d\mu \\
   &\leq \liminf_{k\to\infty} \int (f_k - g_k)\, d\mu \\
   &\leq \liminf_{k\to\infty} \int f_k \, d\mu - \limsup_{k\to\infty} \int g_k \, d\mu \\
   &= \liminf_{k\to\infty} \int f_k \, d\mu - \int g \, d\mu.
   \end{aligned}$$

   Subtracting $\int g \, d\mu (< \infty)$ from both sides of the inequality, we get $\int f \, d\mu \leq \liminf_{k\to\infty} \int f_k \, d\mu$. On the other hand,

   $$\begin{aligned}
   \int (G-f)\, d\mu &= \int \lim_{k\to\infty}(G_k - f_k)\, d\mu \\
   &= \int \liminf_{k\to\infty}(G_k - f_k)\, d\mu \\
   &\leq \liminf_{k\to\infty} \int (G_k - f_k)\, d\mu \\
   &\leq \liminf_{k\to\infty} \int G_k \, d\mu - \limsup_{k\to\infty} \int f_k \, d\mu \\
   &= \int G \, d\mu - \limsup_{k\to\infty} \int f_k \, d\mu.
   \end{aligned}$$

Subtracting $\int G\,d\mu (< \infty)$ from both sides of the inequality we get $\limsup_{k\to\infty} \int f_k\,d\mu \le \int f\,d\mu \le \liminf_{k\to\infty} \int f_k\,d\mu$. Thus, $\int f\,d\mu = \lim_{k\to\infty} \int f_k\,d\mu$ and $\int g\,d\mu \le \int f\,d\mu \le \int G\,d\mu$, hence $\int f\,d\mu$ is finite.

3. (**p.100, exercise 11.4**) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let (g_n) be a sequence of μ-integrable functions on X such that $\sum_{n=1}^{\infty} \int_{|} g_n|\,d\mu < \infty$. Show that $\sum_{n=1}^{\infty} g_n$ is finite μ a.e, and

$$\int \sum_{n=1}^{\infty} g_n\,d\mu = \sum_{n=1}^{\infty} \int g_n\,d\mu.$$

proof (b): By part Corollary 9.9, $\int \sum_{n=1}^{\infty} |g_n|\,d\mu = \sum_{n=1}^{\infty} \int |g_n|\,d\mu < \infty$, hence $\sum_{n=1}^{\infty} |g_n|$ is μ-integrable. We show that $u = \sum_{n=1}^{\infty} |g_n|$ is finite μ a.e. (see also the proof of Corollary 10.13). Let $N = \{x \in X : u(x) = \infty\}$. Then $N = \bigcap_{n=1}^{\infty} \{u \ge n\}$. Since the sequence of measurable sets $\{u \ge n\}$ is decreasing and by the Markov inequality each has finite measure, then $\mu(N) = \lim_{n\to\infty} \mu(\{u \ge n\}) = \lim_{n\to\infty} \frac{1}{n} \int u\,d\mu = 0$. Thus, $u = \sum_{n=1}^{\infty} |g_n|$ is finite μ a.e. Since $|\sum_{n=1}^{\infty} g_n| \le \sum_{n=1}^{\infty} |g_n|$, it follows that $\sum_{n=1}^{\infty} g_n$ is finite μ a.e. Let $h_n = \sum_{m=1}^{n} g_m$, then (h_m) converges to $\sum_{n=1}^{\infty} g_n$ μ a.e. Furthermore, $|h_n| \le \sum_{n=1}^{\infty} |g_n|$, thus by the Dominated Convergence Theorem,

$$\sum_{n=1}^{\infty} \int g_n\,d\mu = \lim_{n\to\infty} \int h_n\,d\mu = \int \lim_{n\to\infty} h_n\,d\mu = \int \sum_{n=1}^{\infty} g_n\,d\mu.$$

4. (**p.100, exercise 11.6**) Give an example of a sequence (u_j) of integrable functions such that $u_j(x) \to u(x)$ for all x where u is an integrable function, but $\lim_{j\to\infty} \int u_j\,d\mu \ne \int u\,d\mu$. Why doesn't this contradict the Lebesgue Dominated Convergence Theorem?

Proof: Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra, and λ the Lebesgue measure. Let $u_j(x) = j\mathbf{1}_{(0,1/j)}(x)$, $j \ge 1$. Clearly, $\lim_{j\to\infty} u_j(x) = 0$ for all $x \in \mathbb{R}$, and

$$\lim_{j\to\infty} \int u_j\,d\lambda = \lim_{j\to\infty} j\lambda((0,1/j)) = \lim_{j\to\infty} j\frac{1}{j} = 1,$$

while

$$\int \lim_{j\to\infty} u_j\,d\lambda = \int 0\,d\lambda = 0.$$

This does not contradict the Lebesgue Dominated Convergence Theorem because the sequence (u_j) is not bounded by an integrable function.

5. (**Exercise 11.7 p. 101**) Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R})$, and suppose $u \in \mathcal{L}^1(\mu)$. Define a function $I : (0,\infty) \to R$ by $I(x) = \int_{(0,x)} u(t)\,d\mu(t) = \int \mathbf{1}_{(0,x)}(t)u(t)\,d\mu(t)$. Show that if μ has no atoms (i.e. $\mu(\{x\}) = 0$ for all x), then I is continuous.

Proof: Suppose μ has no atoms. We will show that $\lim_{x_n \nearrow x} I(x_n) = I(x^-) = I(x^+) = \lim_{y_n \searrow x} I(y_n)$. Note that if $0 < x_n \nearrow x$ and $0 < y_n \searrow x$, then $\lim_{n\to\infty} \mathbf{1}_{(0,x_n)} u =$

$\mathbf{1}_{(0,x)}u$, and $\lim_{n\to\infty}\mathbf{1}_{(0,y_n)}u = \mathbf{1}_{(0,x]}u$. Furthermore, $|\mathbf{1}_{(0,y_n)}u| \le |u|$ and $|\mathbf{1}_{(0,x_n)}u| \le |u|$ and $|u| \in \mathcal{L}^1(\mu)$. By the Lebesgue dominated convergence theorem,

$$\begin{aligned}
I(x^+) - I(x^-) &= \lim_{n\to\infty}\int \mathbf{1}_{(0,x_n)}(t)u(t)\,d\mu(t) - \lim_{n\to\infty}\int \mathbf{1}_{(0,y_n)}(t)u(t)\,d\mu(t) \\
&= \int \mathbf{1}_{(0,x)}(t)u(t)\,d\mu(t) - \int \mathbf{1}_{(0,x])}(t)u(t)\,d\mu(t) \\
&= \int \mathbf{1}_{\{x\}}u(t)d\mu(t) \\
&= u(x)\mu(\{x\}) = 0.
\end{aligned}$$

Thus I is continuous at x for all $x > 0$.