



Measure and Integration solutions of extra problems

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra over \mathbb{R} and λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2^{-k} & \text{if } x \in [k, k+1), k \in \mathbb{Z}, k \geq 0. \end{cases}$$

- (a) Show that f is measurable, i.e. $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$.
(b) Determine the values of $\lambda(\{f > 1\})$, $\lambda(\{f < 1\})$ and $\lambda(\{1/4 \leq f < 1\})$.
(c) Determine the value of $\int f d\lambda$.

Proof(a): It is enough to show that $f^{-1}((-\infty, a]) \in \mathcal{B}_{\mathbb{R}}$ for all $a \in \mathbb{R}$. Now,

$$f^{-1}((-\infty, a]) = \begin{cases} \emptyset & \text{if } a < 0 \\ (-\infty, 0] \cup [k+1, \infty) & \text{if } \frac{1}{2^{k+1}} \leq a < \frac{1}{2^k}, k \geq 0 \\ \mathbb{R} & \text{if } a \geq 1. \end{cases}$$

In all cases one sees that $f^{-1}((-\infty, a]) \in \mathcal{B}_{\mathbb{R}}$. Thus, f is measurable.

Proof(b):

$$\begin{aligned} \lambda(\{f > 1\}) &= \lambda(\emptyset) = 0. \\ \lambda(\{f < 1\}) &= \sum_{k=1}^{\infty} \lambda(\{f = 2^{-k}\}) = \sum_{k=1}^{\infty} \lambda([k, k+1)) = \infty. \\ \lambda(\{1/4 \leq f < 1\}) &= \lambda(\{f = 1/2\}) + \lambda(\{f = 1/4\}) = 2. \end{aligned}$$

Proof(c): Notice that $f = \sum_{k=0}^{\infty} 2^{-k} \mathbf{1}_{[k, k+1)}$. Thus, by Corollary 9.9, f is measurable and

$$\int f d\lambda = \sum_{k=0}^{\infty} \int 2^{-k} \mathbf{1}_{[k, k+1)} d\lambda = \sum_{k=0}^{\infty} 2^{-k} = 2.$$

2. Let (X, \mathcal{B}, μ) be a measure space, and $(G_n)_n \subset \mathcal{B}$ such that $\mu(G_n \cap G_m) = 0$ for $m \neq n$. Show that $\mu(\bigcup_n G_n) = \sum_n \mu(G_n)$.

Proof: Let $A_1 = G_1$, $B_1 = \emptyset$. For $n \geq 2$, set $A_n = G_n \setminus \bigcup_{m=1}^{n-1} G_m$ and $B_n = G_n \cap \bigcup_{m=1}^{n-1} G_m = \bigcup_{m=1}^{n-1} (G_n \cap G_m)$. Then,

– $G_n = A_n \cup B_n$ for all $n \geq 1$,

– $A_n \cap A_m = \emptyset$ for $m \neq n$,

– $\mu(B_n) = 0$ for all $n \geq 1$ (since $\mu(G_n \cap G_m) = 0$ for $n \neq m$), hence $\mu(G_n) = \mu(A_n)$ for all $n \geq 1$,

– $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} G_n$: clearly the left handside is a subset of the right handside. Now, let $x \in \bigcup_{n=1}^{\infty} G_n$, then $x \in G_n$ for some n . Let n_0 be the smallest positive integer such that $x \in G_{n_0}$, then $x \in A_{n_0} \subseteq \bigcup_{n=1}^{\infty} A_n$.

Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} G_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(G_n).$$

3. Let (X, \mathcal{B}, ν) be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on \mathcal{B} by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n} \nu(B \cap E_n) / (\nu(E_n) + 1)$.

(a) Prove that μ is a finite measure on (X, \mathcal{B}) .

(b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ if and only if $\nu(B) = 0$.

Proof (a): Clearly $\mu(\emptyset) = 0$, and

$$\mu(X) = \sum_{n=1}^{\infty} 2^{-n} \nu(E_n) / (\nu(E_n) + 1) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

Now, let (C_n) be a disjoint sequence in \mathcal{B} . Then,

$$\begin{aligned} \mu\left(\bigcup_{m=1}^{\infty} C_m\right) &= \sum_{n=1}^{\infty} 2^{-n} \nu\left(\left(\bigcup_{m=1}^{\infty} C_m\right) \cap E_n\right) / (\nu(E_n) + 1) \\ &= \sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{\infty} \nu(C_m \cap E_n) / (\nu(E_n) + 1) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-n} \nu(C_m \cap E_n) / (\nu(E_n) + 1) \\ &= \sum_{m=1}^{\infty} \mu(C_m). \end{aligned}$$

Thus, μ is a finite measure.

Proof (b): Suppose that $\nu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n , hence $\mu(B) = 0$.

Conversely, suppose $\mu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n . Since $X = \bigcup_{n=1}^{\infty} E_n$

(disjoint union), then

$$\nu(B) = \nu(B \cap \bigcup_{n=1}^{\infty} E_n) = \nu(\bigcup_{n=1}^{\infty} (B \cap E_n)) = \sum_{n=1}^{\infty} \nu(B \cap E_n) = 0.$$

4. Let (E, \mathcal{B}, μ) be a measure space, and $\overline{\mathcal{B}}^\mu$ be the completion of the σ -algebra \mathcal{B} with respect to the measure μ (see exercise 4.13, p.29). We denote by $\overline{\mu}$ the extension of the measure μ to the σ -algebra $\overline{\mathcal{B}}^\mu$. Suppose $f : E \rightarrow E$ is a function such that $f^{-1}(B) \in \mathcal{B}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{B}}^\mu$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{B}}^\mu$.

Proof: Let $\overline{B} \in \overline{\mathcal{B}}^\mu$, then there exist $A, B \in \mathcal{B}$ such that $A \subseteq \overline{B} \subseteq B$, $\mu(B \setminus A) = 0$ and $\overline{\mu}(\overline{B}) = \mu(A)$. Then, $f^{-1}(A), f^{-1}(B) \in \mathcal{B}$ satisfy $f^{-1}(A) \subseteq f^{-1}(\overline{B}) \subseteq f^{-1}(B)$ and $\mu(f^{-1}(B) \setminus f^{-1}(A)) = \mu(f^{-1}(B \setminus A)) = \mu(B \setminus A) = 0$. Thus, $f^{-1}(\overline{B}) \in \overline{\mathcal{B}}^\mu$ and $\overline{\mu}(f^{-1}(\overline{B})) = \mu(f^{-1}(A)) = \mu(A) = \overline{\mu}(\overline{B})$.

5. Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ -algebra over X containing \mathcal{C} , and let \mathcal{D} be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.

- (a) Show that \mathcal{D} is a σ -algebra over X .
(b) Show that $\mathcal{D} = \sigma(\mathcal{C})$.

Proof (a): Clearly $\emptyset \in \mathcal{D}$ since \emptyset belongs to every σ -algebra. Let $A \in \mathcal{D}$, then there is a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $A \in \sigma(\mathcal{C}_0)$. But then $A^c \in \sigma(\mathcal{C}_0)$, hence $A^c \in \mathcal{D}$. Finally, let $\{A_n\}$ be in \mathcal{D} , then for each n there exists a countable collection $\mathcal{C}_n \subseteq \mathcal{C}$ such that $A_n \in \sigma(\mathcal{C}_n)$. Let $\mathcal{C}_0 = \bigcup_n \mathcal{C}_n$, then $\mathcal{C}_0 \subseteq \mathcal{C}$, and \mathcal{C}_0 is countable. Furthermore, $\sigma(\mathcal{C}_n) \subseteq \sigma(\mathcal{C}_0)$, and hence $A_n \in \sigma(\mathcal{C}_0)$ for each n which implies that $\bigcup_n A_n \in \sigma(\mathcal{C}_0)$. Therefore, $\bigcup_n A_n \in \mathcal{D}$ and \mathcal{D} is a σ -algebra.

Proof (b): By definition $\mathcal{D} \subseteq \sigma(\mathcal{C})$. Also, $\mathcal{C} \subseteq \mathcal{D}$ since $C \in \sigma(\{C\})$ for every $C \in \mathcal{C}$. Since $\sigma(\mathcal{C})$ is the smallest σ -algebra over X containing \mathcal{C} , then by part (a) $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \sigma(\mathcal{C})$.

6. Let (X, \mathcal{B}, μ) be a probability space, i.e. $\mu(X) = 1$. Let $f : X \rightarrow [0, 1)$ be a measurable function such that $\mu\left(f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)\right) = \frac{1}{2^n}$ for $n \geq 1$ and $k = 0, 1, \dots, 2^n - 1$.

Show that $\int_X f^2 d\mu = \frac{1}{3}$.

Proof Let $A_{k,n} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$, for $n \geq 1$ and $k = 0, 1, \dots, 2^n - 1$. For $n \geq 1$,

let $g_n = \sum_{k=0}^{2^n-1} \frac{k^2}{4^n} 1_{A_{k,n}}$. Then, g_n is a sequence of non-negative measurable functions such that $g_n(x) \uparrow f^2(x)$ for all $x \in [0, 1)$. Furthermore,

$$\int_X g_n d\mu = \sum_{k=0}^{2^n-1} \frac{k^2}{8^n} = \frac{2^n(2^n-1)(2^{n+1}-1)}{6 \cdot 8^n}.$$

By Beppo-Levi,

$$\int_X f^2 d\mu = \lim_{n \rightarrow \infty} \int_E g_n d\mu = \lim_{n \rightarrow \infty} \frac{2^n(2^n - 1)(2^{n+1} - 1)}{6 \cdot 8^n} = \frac{1}{3}.$$