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Measure and Integration Solutions to More Extra Exercises Final 2008

1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f \in \mathcal{L}^p(\mu)$ be such that $f = \mathcal{L}^p(\mu) - \lim_{n\to\infty} f_n$. Prove that for every $\epsilon > 0$, one has

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

Proof: Since $f = \mathcal{L}^p(\mu) - \lim_{n\to\infty} f_n$, then $\lim_{n\to\infty} ||f_n - f||_p = 0$. Given any $\epsilon > 0$, by Markov inequality

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = \mu(\{x : |f_n(x) - f(x)|^p \geq \epsilon^p\}) \leq \frac{1}{\epsilon^p}||f_n - f||_p^p.$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

2. Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$) and let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that $\int_X |f_n| d\mu = \alpha$ for all $n \geq 1$, where $\alpha \in (0, \infty)$. Let

$$A_n = \{x : |f_n(x) - \int_X f_n d\mu| \geq n^2\}.$$

Show that $\mu(\limsup_{n\to\infty} A_n) = 0$. (Hint: use Borel-Cantelli Lemma).

Proof By Markov Inequality we have

$$\mu(A_n) \leq \frac{1}{n^2} \int_X |f_n(x) - \int_X f_n d\mu| \leq \frac{2\alpha}{n^2}.$$

Since $\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{2\alpha}{n^2} < \infty$, it follows by Borel-Cantelli Lemma (Exercise 6.9) that $\mu(\limsup_{n\to\infty} A_n) = 0$.

3. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$.

   (a) Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$.

   (b) Prove that if ν is a finite measure, then $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\mu)$.

Proof Suppose first that $f = 1_A$ is the indicator function of some set $A \in \mathcal{B}$. Then

$$\int_E f\,d\mu = \mu(A) \le \nu(A) = \int_E f\,d\nu.$$

Suppose now that $f = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

$$\int_E f\,d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \le \sum_{k=1}^{n} \alpha_k \nu(A_k) = \int_E f\,d\nu.$$

Finally, let f be a non-negative measurable function, then there exists a sequence of non-negative measurable step functions f_n such that $f_n \uparrow f$. By the Monotone Convergence Theorem,

$$\int_E f\,d\mu = \lim_{n\to\infty} \int_E f_n\,d\mu \le \lim_{n\to\infty} \int_E f_n\,d\nu = \int_E f\,d\nu.$$

The above implies that if $f \in L^1(\nu)$, then $f \in L^1(\mu)$, i.e. $L^1(\nu) \subseteq L^1(\mu)$.

If ν is a finite measure, then by exercise 12.1 (ii) and the above, we have $L^2(\nu) \subseteq L^1(\nu) \subseteq L^1(\mu)$.

4. Consider the measure space $([0,1), \mathcal{B}([0,1), \lambda)$, where $\mathcal{B}([0,1))$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure on $[0,1)$. Define a sequence of measurable functions f_n on $[0,1)$ as follows: given $n \ge 1$, there exist an $m \ge 0$ and $0 \le l \le 2^m - 1$ such that $n = 2^m + l$ (note that this representation is unique). Set $f_n = f_{2^m+l} = 1_{[l/2^m,(l+1)/2^m)}$.
   (a) Determine explicitly $f_1, f_2, f_3, f_4, f_5, f_6, f_7$.
   (b) Show that $\limsup_{n\to\infty} f_n(x) = 1$ for all $x \in [0,1]$.
   (c) Show that $\lim_{n\to\infty} ||f_n||_1 = 0$. Conclude that L^1-convergence does not imply λ a.e. convergence.

Proof (a) Notice that $1 = 2^0 + 0$, $2 = 2^1 + 0$, $3 = 2^1 + 1$, $4 = 2^2 + 0$, $5 = 2^2 + 1$, $6 = 2^2 + 2$, $7 = 2^2 + 3$. Thus, $f_1 = 1_{[0,1)}$, $f_2 = 1_{[0,1/2)}$, $f_3 = 1_{[1/2,1)}$, $f_4 = 1_{[0,1/4)}$, $f_5 = 1_{[1/4,2/4)}$, $f_6 = 1_{[2/4,3/4)}$, $f_7 = 1_{[3/4,1)}$.

Proof (b) Notice that for each $m \ge 1$, $\{[l/2^m, (l+1)/2^m) : 0 \le l \le 2^m - 1\}$ forms a partition of $[0,1)$. Hence, for each $x \in [0,1)$ and for every $m \ge 0$ there exists an $0 \le l \le 2^m - 1$ such that $f_{2^m+l}(x) = 1$. Thus $f_n(x) = 1$ for infinitely many n. This shows that $\limsup_{n\to\infty} f_n(x) = 1$ for all $x \in [0,1]$.

Proof (c) If $n = 2^m + l$, then $||f_n||_{L^1(\lambda_{[0,1)})} = \int_{[0,1)} f_{2^m+l} d\lambda_{[0,1)} = 1/2^m$. Since $m \to \infty$ as $n \to \infty$, taking limits we see that $\lim_{n\to\infty} ||f||_{L^1(\lambda_{[0,1)})} = 0$. This shows that $f_n \to 0$ in $L^1(\lambda_{[0,1)})$ but from part (b), f_n **does not** converge to 0 μ a.e.

5. Let $E_1 = E_2 = \mathbb{N} = \{1, 2, 3, \cdots\}$. Let $\mathcal{B}$ be the collection of all subsets of $\mathbb{N}$. and $\mu_1 = \mu_2$ be counting measure on $\mathbb{N}$. Let $f : E_1 \times E_2 \to \mathbb{R}$ by $f(n, n) = n$, $f(n, n+1) = -n$ and $f(n, m) = 0$ for $m \neq n, n+1$.

   (a) Prove that $\int_{E_1} \int_{E_2} f(n, m) d\mu_2(m) d\mu_1(n) = 0$.

   (b) Prove that $\int_{E_2} \int_{E_1} f(n, m) d\mu_1(n) d\mu_2(m) = \infty$.

   (c) Explain why parts (a) and (b) do not contradict Fubini's Theorem.

Proof (a) For each fixed n one has

$$\int_{E_2} f(n, m) d\mu_2(m) = f(n, n)\mu_2(\{n\}) + f(n, n+1)\mu_2(\{n+1\}) = 0.$$

Thus, $\int_{E_1} \int_{E_2} f(n, m) d\mu_2(m) d\mu_1(n) = 0$.

Proof (b) For each fixed m,

$$\int_{E_1} f(n, m) d\mu_1(n) = f(m, m)\mu_1(\{m\}) + f(m-1, m)\mu_1(\{m-1\}) = 1.$$

Thus,$\int_{E_2} \int_{E_1} f(n, m) d\mu_1(n) d\mu_2(m) = \int_{E_2} 1 d\mu_2(m) = \mu_2(E_2) = \infty$.

Proof (c) Parts (a) and (b) do not contradict Fubini's Theorem because the function f is not $\mu_1 \times \mu_2$ integrable. This follows from

$$\int_{E_1} \int_{E_2} |f(n, m)| d\mu_2(m) d\mu_1(n) = \int_{E_1} 2n d\mu_1(n) = \sum_{n=1}^{\infty} 2n = \infty.$$