Solutions to Exercises in Extra Lecture Notes, SCI 113 Spring 2008

(1) **Exercise 1** We need to verify properties (1)-(10) of a vector space (p.1 of extra LN).

(1) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$ and $\mathbf{v} = \begin{pmatrix} d & e \\ f & 0 \end{pmatrix} \in M$, then $\mathbf{u}+\mathbf{v} = \begin{pmatrix} a+d & b+e \\ c+f & 0 \end{pmatrix} \in M$.

(2) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$ and $\mathbf{v} = \begin{pmatrix} d & e \\ f & 0 \end{pmatrix} \in M$, then

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} a+d & b+e \\ c+f & 0 \end{pmatrix} = \begin{pmatrix} d+a & e+b \\ f+c & 0 \end{pmatrix} = \mathbf{v}+\mathbf{u}.$$

(3) If $\mathbf{u}$, $\mathbf{v}$ as above and $\mathbf{w} = \begin{pmatrix} g & h \\ i & 0 \end{pmatrix}$, then

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = \begin{pmatrix} a+(d+g) & b+(e+h) \\ c+(f+i) & 0 \end{pmatrix} = \begin{pmatrix} (a+d)+g & (b+e)+h \\ (c+f)+i & 0 \end{pmatrix} = (\mathbf{u}+\mathbf{v})+\mathbf{w}.$$

(4) The zero vector is given by $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M$, notice that

$$\mathbf{u}+\mathbf{0} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \mathbf{u}.$$

(5) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$, then the vector $-\mathbf{u} = \begin{pmatrix} -a & -b \\ -c & 0 \end{pmatrix}$ has the property that $\mathbf{u} + -\mathbf{u} = \mathbf{0}$.

(6) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$, and λ is a real number, then $\lambda\mathbf{u} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & 0 \end{pmatrix} \in M$.

(7) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$ and $\mathbf{v} = \begin{pmatrix} d & e \\ f & 0 \end{pmatrix} \in M$, and λ is a real number, then

$$\lambda(\mathbf{u}+\mathbf{v}) = \begin{pmatrix} \lambda(a+d) & \lambda(b+e) \\ \lambda(c+f) & 0 \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & 0 \end{pmatrix} + \begin{pmatrix} \lambda d & \lambda e \\ \lambda f & 0 \end{pmatrix} = \lambda\mathbf{u}+\lambda\mathbf{v}.$$

(8) if α, β are real numbers, and $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$, then

$$(\alpha+\beta)\mathbf{u} = \mathbf{u} = \begin{pmatrix} (\alpha+\beta)a & (\alpha+\beta)b \\ (\alpha+\beta)c & 0 \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & 0 \end{pmatrix} + \begin{pmatrix} \beta a & \beta b \\ \beta c & 0 \end{pmatrix} = \alpha\mathbf{u}+\beta\mathbf{u}.$$

(9) if α, β are real numbers, and $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$, then

$$\alpha(\beta\mathbf{u}) = \alpha \begin{pmatrix} \beta a & \beta b \\ \beta c & 0 \end{pmatrix} = \alpha\beta \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = (\alpha\beta)\mathbf{u}.$$

(10) If $\mathbf{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in M$, then it is clear that

$$1\mathbf{u} = \mathbf{u}.$$

Hence M is a vector space under the the usual operations of addition and scalar multiplication of matrices.

(2) **Exercise 2** We want to write $\mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ in $\mathbb{R}^3$ as a linear combination of $\mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. In other words, we want to find real numbers c_1, c_2, c_3 such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}.$$

This leads to

$$\begin{cases} -c_2 + 3c_3 = -1 \\ c_1 + c_2 + c_3 = -2 \\ 4c_1 + 2c_2 + 2c_3 = -2. \end{cases}$$

Solving this system, we get that $c_1 = 1$, $c_2 = -2$, and $c_3 = -1$. Thus,

$$\mathbf{w} = \mathbf{v_1} - 2\mathbf{v_2} - \mathbf{v_3}.$$

(3) **Exercise 3** To show that S is a spanning set for $\mathbb{R}^3$, we need to prove that any vector $\mathbf{u}$ of $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{v_1} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$. So suppose $\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, we need to find real numbers c_1, c_2, c_3 so that $\mathbf{w} = \mathbf{v_1} - 2\mathbf{v_2} - \mathbf{v_3}$. This vector equation leads to the following system of linear equations (in the variables c_1, c_2, and c_3)

$$\begin{cases} 4c_1 - c_2 + 2c_3 = a \\ 7c_1 + 2c_2 - 3c_3 = b \\ 3c_1 + 6c_2 + 5c_3 = c. \end{cases}$$

This system has a unique solution since the matrix $A = \begin{pmatrix} 4 & -1 & 2 \\ 7 & 2 & -3 \\ 3 & 6 & 5 \end{pmatrix}$ has a non-zero determinant ($\det(A) = 228$). The unique solution is given by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{228} \begin{pmatrix} 28 & 17 & -1 \\ -44 & 14 & 26 \\ 36 & -27 & 15 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Hence, $\mathbf{w}$ can be written as a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$. So S is a spanning set.

(4) **Exercise 4** To show that the set $S = \{x^2-1, 2x+5\}$ is linearly independent in P_2, we need to show that if $c_1(x^2-1)+c_2(2x+5) = 0$, then $c_1 = c_2 = 0$.

So assume $c_1(x^2-1)+c_2(2x+5)=0$. Rewriting we get $c_1x^2+2c_2x+(-c_1+5c_2)=0$. Thus

$$\begin{cases} c_1 = 0 \\ 2c_2 = 0 \\ -c_1+5c_2 = 0. \end{cases}$$

This system has a unique solution $c_1=c_2=0$. So S is linearly independent.

(5) **Exercise 5** To show that the set $S=\left\{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right\}$ forms a basis for $M_{2,2}$, we need two verify that S is linearly independent, and that S is a spanning set. We begin with linear independence. Suppose

$$c_1\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}+c_2\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}+c_3\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}+c_4\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This lead to

$$\begin{cases} 2c_1+c_2 = 0 \\ 4c_2+c_3+c_4 = 0 \\ 3c_3+2c_4 = 0 \\ 3c_1+c_2+2c_3 = 0. \end{cases}$$

This system has a unique solution $c_1=c_2=c_3=c_4=0$. Thus, S is linearly independent. To show S is a spanning set, we need to show that any 2×2 matrix is a linear combination of elements of S. So given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we need to find real numbers c_1, c_2, c_3, and c_4 such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}=c_1\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}+c_2\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}+c_3\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}+c_4\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

This is equivalent to

$$\begin{cases} 2c_1+c_2 = a \\ 4c_2+c_3+c_4 = b \\ 3c_3+2c_4 = c \\ 3c_1+c_2+2c_3 = d. \end{cases}$$

Since the matrix $A=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 3 & 1 & 2 & 0 \end{pmatrix}$ has a non-zero determinant, the system has a unique solution given by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}=A^{-1}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Thus S is a spanning set.

(6) **Exercise 6** To evaluate $T\left(\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\right)$, we first need to write the matrix $\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$ as a linear combination of elements of

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

as follows

$$\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since T is a linear transformation, then

$$\begin{aligned} T\left(\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\right) &= T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + 3T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) - T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) + 4T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + 3\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} + 4\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 12 & -1 \\ 7 & 4 \end{pmatrix}. \end{aligned}$$

(7) **Exercise 7** To find the standard matrix A of the linear transformation T, we calculate

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 13 \\ 6 \end{pmatrix},$$

$$T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -9 \\ 5 \end{pmatrix},$$

and

$$T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Hence, the standard matrix A is given by

$$A = \begin{pmatrix} 13 & -9 & 4 \\ 6 & 5 & -3 \end{pmatrix}.$$

(8) **Exercise 8** We find the images of the elements of B, and we express them as linear combinations of elements of B'. Now

$$T(1) = \int_0^x t^0\,dt = x = 0(1) + 1(x) + 0(x^2) + 0(x^3) + 0(x^4),$$

$$T(x) = \int_0^x t^1\,dt = \frac{x^2}{2} = 0(1) + 0(x) + \frac{1}{2}(x^2) + 0(x^3) + 0(x^4),$$

$$T(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = 0(1) + 0(x) + 0(x^2) + \frac{1}{3}x^3 + 0(x^4),$$

$$T(x^3) = \int_0^x t^3\,dt = \frac{x^4}{4} = 0(1) + 0(x) + 0(x^2) + 0(x^3) + \frac{1}{4}(x^4).$$

Thus, the required matrix is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}.$$