Solutions Test 2, SCI 113 Spring 2008

- Write your name and student number on each page you hand in.
- You are allowed to use the book Mathematical Techniques by Jordan and Smith and the lecture notes by Frits Beukers.
- You should explain how you have calculated your answers.

(1) (1pt) Find the matrix representation for the orthogonal reflection in the plane in the line $y = -x$.

Solution: The matrix A representing the orthogonal reflection in the line $y = -x$ is a 2×2 matrix whose columns are the coordinates of the orthogonal reflection in the line $y = x$ of the basis unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$. It is easiy seen that the orthogonal reflection of $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and the orthogonal reflection of $\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Thus, the required matrix is given by

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

(2) (a) (1 pt) Using only the properties of determinants (and without directly evaluating the determinants), show that

$$\begin{vmatrix} 100 & 100 & 100 \\ 200 & 203 & 208 \\ 100 & 101 & 102 \end{vmatrix} = 100 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix}.$$

Solution (a):

$$\begin{vmatrix} 100 & 100 & 100 \\ 200 & 203 & 208 \\ 100 & 101 & 102 \end{vmatrix} \overset{-c_2 + c_3 \to c_3}{=} \begin{vmatrix} 100 & 100 & 0 \\ 200 & 203 & 5 \\ 100 & 101 & 1 \end{vmatrix}$$

$$\overset{-c_1 + c_2 \to c_2}{=} \begin{vmatrix} 100 & 0 & 0 \\ 200 & 3 & 5 \\ 100 & 1 & 1 \end{vmatrix}$$

$$\overset{-c_1 + c_2 \to c_2}{=} 100 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix}.$$

(b) (1 pt) Show that

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b).$$

Solution (b):

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} \overset{c_1+c_2+c_3 \to c_1}{=} \begin{vmatrix} 3a+b & 3a+b & 3a+b \\ a & a+b & a \\ a & a & a+b \end{vmatrix}$$

$$= (3a+b)\begin{vmatrix} 1 & 1 & 1 \\ a & a+b & a \\ a & a & a+b \end{vmatrix}$$

$$\overset{-c_1+c_2 \to c_2}{=} (3a+b)\begin{vmatrix} 1 & 0 & 1 \\ a & b & a \\ a & 0 & a+b \end{vmatrix}$$

$$\overset{-c_1+c_3 \to c_3}{=} (3a+b)\begin{vmatrix} 1 & 0 & 0 \\ a & b & 0 \\ a & 0 & b \end{vmatrix} = (3a+b)b^2.$$

(3) Let $A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$.

(a) (1 pt) Find the determinant of the matrix A. **Solution (a)**: $\det(A) = 3$.

(b) (1.5 pt) Does the inverse matrix A^{-1} exist? If yes, find A^{-1}.

Solution (b): A^{-1} exists since $\det(A) \neq 0$, and $A^{-1} = \begin{pmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{pmatrix}$.

(c) (1 pt) Determine the unique solution of the system

$$\begin{cases} -x+3y+2z = 2 \\ -2y+z = 3 \\ x-2z = 5. \end{cases}$$

Solution (c): The given system is equivalent to the matrix equation $AX = B$, where A is the given matrix, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The unique solution is given by $X = A^{-1}B = \begin{pmatrix} 61/3 \\ 7/3 \\ 23/3 \end{pmatrix}$.

(4) Let

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ 3 & 0 & -3 \end{pmatrix}.$$

(a) (2.5 pt) Find the eigenvalues of A and their corresponding eigenvectors.

Solution (a): First we determine the eigenvalues by solving the equation $\det(A - \lambda I_3) = 0$. Calculating we see that

$$det(A - \lambda I_3) = \lambda(2-\lambda)(\lambda+1) = 0$$

Thus, $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = -1$. The corresponding eigenvalues are given by

$t\begin{pmatrix}1\\1\\1\end{pmatrix}$ (corresponding to $\lambda_1 = 0$)

$t\begin{pmatrix}0\\1\\0\end{pmatrix}$ (corresponding to $\lambda_2 = 2$)

$t\begin{pmatrix}2\\2\\3\end{pmatrix}$ (corresponding to $\lambda_3 = -1$).

(b) (1 pt) Let $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}2\\2\\3\end{pmatrix}$. Show that v_1, v_2 and v_3 are linearly independent, i.e. show that if $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Solution (b): Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$, this leads to

$$\begin{cases}\alpha_1 + 2\alpha_3 = 0\\ \alpha_1 + \alpha_2 + 2\alpha_3 = 0\\ \alpha_1 + 3\alpha_3 = 0.\end{cases}$$

Solving this system gives a unique solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$.