The Search for a Cube Puzzle

by Marc van Kreveld

The puzzle idea and the pieces

Needless to say that the cube is a well-known starting point for puzzles. For the puzzle I had in mind, not a whole cube, but only its corners and edges were needed. Each piece consists of one corner and three adjacent *half-edges*. Figure 1 shows the cube puzzle when it is nearly solved. Clearly, there are eight pieces in this puzzle. The question is which pieces make an interesting set, which you can interpret as: a difficult puzzle.

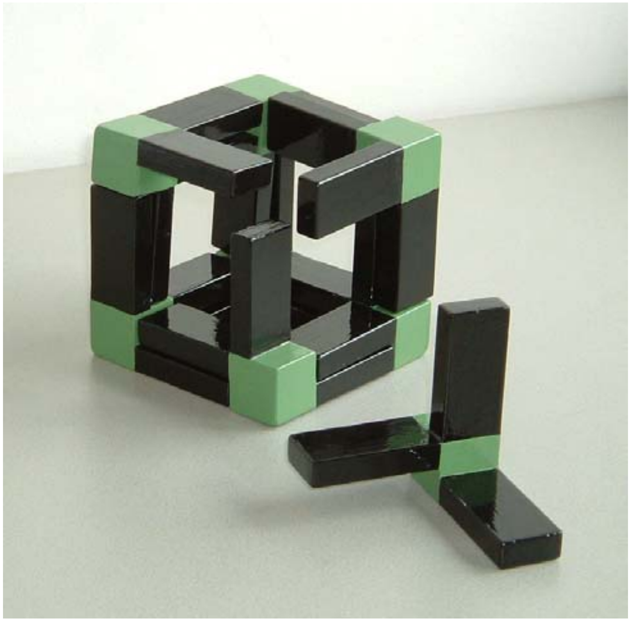

Figure 1. The cube puzzle with one piece not placed yet

To examine this question further let us consider one piece. When we place it before us as in Figure 2, the three half-edges are pointing towards us. Each of the three half-edges can be attached in four different ways to the corner. Consider the half-edge left and below. It is the *top half* of the whole edge; we call such a half-edge of type A. The half-edge could also have been the *right half* (B), the *bottom half* (C) or the *left half* (D) of the edge. Note that these type names are valid only for the bottom left half-edge, and only if it is in the specified position. If you examine the half-edge types carefully, you will notice that a half-edge of type A only fits with one of type D to make a whole edge, and a half-edge of type B only fits with one of type C.

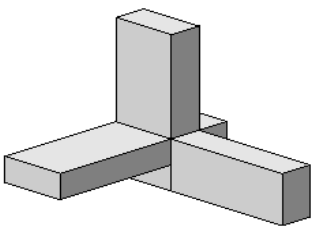

Figure 2. One piece

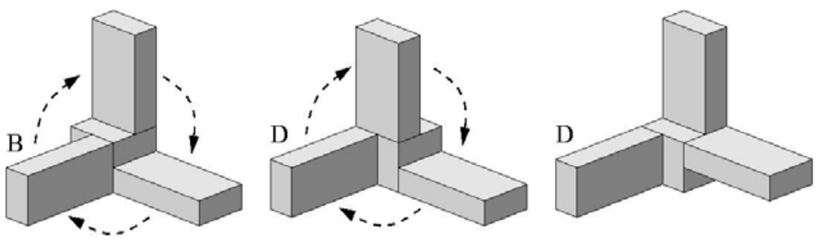

Figure 3. Naming scheme of a piece BDD

Let us next consider a whole piece (Figure 2). The bottom left half-edge is of type B. When we rotate the whole piece as indicated in Figure 3, we can determine the type of the second half-edge: type D. When we rotate it again we note that the third half-edge is also of type D. We will call this piece of type BDD. Note that types DBD and DDB are the same piece; we will always use the lexicographically smallest name. In total, there appear to be 24 different pieces: AAA, BBB, CCC, AAB, AAC, AAD, ABB,

BBC, BBD, ACC, BCC, CCD, ADD, BDD, CDD, ABC, ABD, ACB, ACD, ADB, ADC, BCD, and BDC. Note that ABC and ACB are not the same!

The requirements

As there are 24 different pieces, and we need 8 to make a puzzle, potentially there are 24^8 different puzzles. But of course, every good set must have pieces with in total an equal number of A's and D's, and an equal number of B's and C's. Since there are 24 half-edges, we prefer to have 6 half-edges of each type. We also prefer to not use the pieces of types AAA, BBB, CCC, or DDD. These pieces are rotationally symmetric, and therefore they make the puzzle easier. As a third requirement, we want all 8 pieces to be different. Finally, we would like a puzzle that has only one, unique solution for its assembly. The question is whether a puzzle exists that fulfils those requirements. Counting again without AAA, BBB, CCC, and DDD, we choose 20 sets of 8 pieces, which generates 125,970 sets of 8 different pieces.

To solve the question whether a good puzzle exists we call in the help of a computer. But instead of writing a puzzle solver that examines all of these 125,970 sets, and try all 11,022,480 ways in which they might fit (11,022,480 = 7! x 3^7), I chose a different approach. Consider any solved puzzle. We can cut its 12 edges into half-edges in four different ways, as shown in Figure 4. Once we choose how to cut, the 8 pieces of the puzzle are determined, and we have a set of pieces that has at least one solution. We will consider all 4^{12} ways to cut all 12 edges into two. Then we have generated all 16,777,216 solutions to all puzzles that have some solution. Having done that, we group the solutions if they create the same set of 8 pieces. If there is some solution that generates a specific set of 8 pieces, such that no other solution gives the same 8 pieces, then that set of 8 pieces has a unique solution.

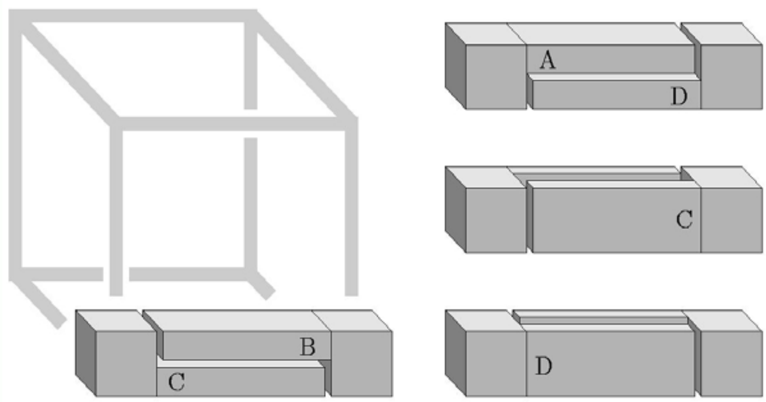

Figure 4. Four types of connections in a solution

The results

It does not take long to implement these ideas. The 16,777,216 solutions are generated by a program with 12 nested for-loops, and the 8 resulting pieces are tested for the requirements. Only 1,023,360 of the solutions give pieces that do not use AAA, etc., have no equal pieces, and have 6 half-edges of each type. These are sorted to group equal sets of 8 pieces. It appears that there are 2290 different puzzles. The most difficult ones have 24 solutions and the easiest ones have 1656 solutions. Every set that appears, appears a multiple of 24 times. This is logical: a cube has 24 symmetries, and we find all 24 symmetrical solutions too. So the puzzles with 24 solutions actually are uniquely solvable. There are 34 such puzzles, for example: AAD, ACB, ACD, ADB, ADD, BBC, BCC, BDC. This is the puzzle that I made and that is shown in Figures 1 and 5.

Equivalence of puzzles

I was happy with these results for a while, but then I wondered if the 34 uniquely solvable puzzles were really equally difficult. In fact, some puzzles are even essentially the same. Consider the puzzles:

AAD, ABC, ACD, ADB, ADD, BBC, BCC, BCD, and
AAD, ACB, ABD, ADC, ADD, BCC, BBC, BDC.

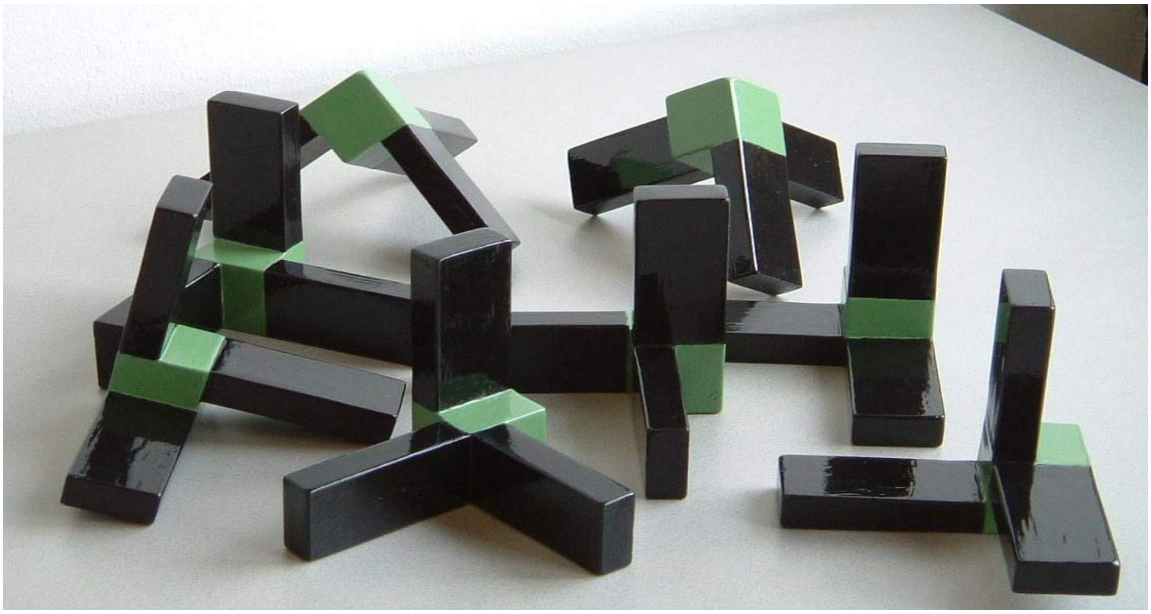

Figure 5. Eight pieces of a uniquely solvable puzzle

If in the first puzzle we replace every B by a C, and every C by a B, then we get the second puzzle. Similarly, if we would have replaced every A with a D and every D with an A, we also get a puzzle with a unique solution. Or: replace every A with a B, every D with a C, every B with a D, and every C with an A. Such a simultaneous replacement is called *an equivalence*, in correspondence with mathematical terminology. Two puzzles that can be transformed into each other are called equivalent. There appear to be 8 equivalences, and therefore groups of 8 puzzles should be essentially the same. At first it seems strange that there were 34 different uniquely solvable puzzles, since 34 is not divisible by 8. This can be explained as follows. There are two puzzles, namely:

AAD, ABC, ACD, ADB, ADD, BBC, BCC, BDC, and
AAD, ABD, ACB, ADC, ADD, BBC, BCC, BCD,

for which all 8 equivalences transform the puzzles into each other or into themselves. So even though the simultaneous replacement is different, you may still get the same set of pieces. The other puzzles appear in groups of 8 equivalent puzzles. So there are 5 equivalence classes on essentially different puzzles.

A definition of difficulty

Anyone who would try to solve a puzzle of the type presented here will probably start with a base, four pieces that form the bottom of the cube (Figure 6). These are the ones that are supported by the table. After this bottom loop is built, the puzzler will continue to add the four top pieces. If this works, fine, and otherwise, a new base will be formed. Probably, if the base is part of the solution, the puzzler will be able to complete the solution. After all, there are only four pieces left. This led me to the following definition of difficulty: the ratio of the number of loops in a correct solution and the number of loops that can be formed in total with the 8 pieces. For completeness: a loop is a set of 4 pieces that forms one side of the cube, with 4 complete edges. In fact, the difficulty is defined as the probability that a random loop can be completed to a solution. To determine the number of loops for each set of 8 pieces, the computer was set to work again. The 5 equivalence classes of puzzles have 107, 116, 118, 118, and 122 loops (and indeed, equivalent puzzles have the same number of loops). The puzzle shown before is actually one of those easiest uniquely solvable puzzles, where the chances of getting a good loop in the first try are 6/107. With the most difficult puzzle, the chances are worse than 1/20. Here is the code for one of these sets: AAB, AAD, ABC, ADD, BBC, BCC, BDC, CDD.

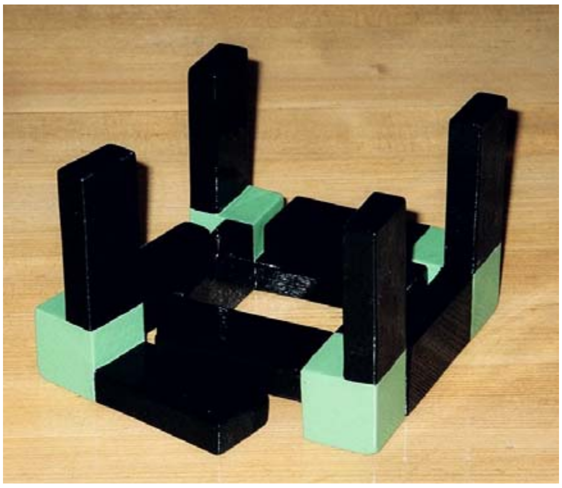

Figure 6. Base with four pieces

Different appearances, similar puzzles

The idea for the cube puzzle presented here can give rise to several different visually different puzzles. For example, the half-edges could have been triangular (right-angled, equiangular). Also, the puzzle shown in Figure 7 is the same, but here the corners have been made very thick, and the half-edges are replaced by a peg-and-hole connection. It is one of the easiest four puzzles with 69 solutions (24 x 69 = 1656). It is also possible to design a puzzle with 12 pieces, one for each edge. Since at each corner of a cube three edges meet, the ends of the edges must be designed so that three edge end types can form a corner. There are several choices for such puzzles, which are left to the interested reader.

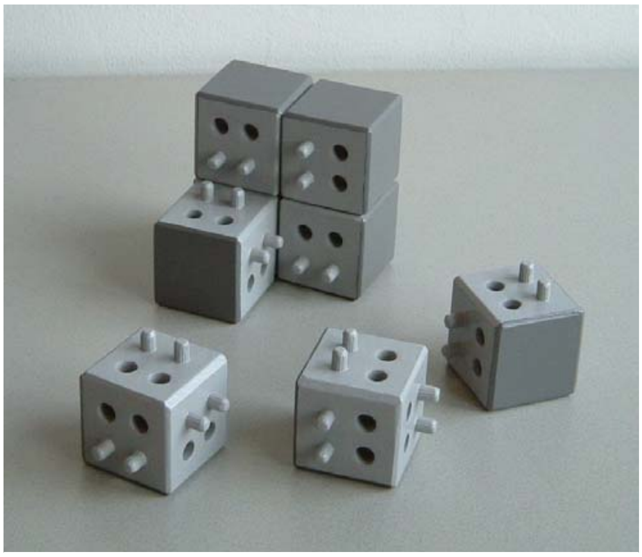

Figure 7. Cube puzzle with peg-and-hole connections

The well-known wirrel-warrel cube [1] is in fact a puzzle of a similar type too. It has a piece for every side of the cube, and the pieces must be made to fit along edges and in the corners. Edges are subdivided into 5 unit cubes and are attached to one of the two incident sides. The corners are one unit cube. Different attachments give the different puzzle pieces.

Reference:

[1] *Happy Cubes (wirrel-warrel),* CFF 50, part 4/6, October 1999.