Algorithmic Modeling and Complexity **Fall 2003**

Lecture 9: 30 September

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9.1 Overview

Many issues of algorithmic modeling arise in *planning*, i.e. in assigning sufficient resources to tasks so these tasks can be carried out. We consider a classical example known as the *transportation problem*, originally due to Hitchcock [2] and others and analyzed in the framework of network modeling and LP by e.g. Ford and Fulkerson [1] and Dantzig. The problem will be shown to be equivalent to the modeling framework of *minimum cost flow*. We will also consider the *assignment problem* as a special case. It will be shown to be equivalent to the *single-pair shortest path problem*.

9.2 The transportation problem

The transportation problem is the following problem:

> Given are m depots with quantities $a_1, \ldots, a_m$ of a certain commodity in store, and n customers which require quantities $b_1, \ldots, b_n$ respectively. The cost of transporting 1 unit of the commodity from depot i to customer j is $c_{ij} \geq 0$ and fractional amounts take a proportional cost. Determine the quantity of the goods that must be transported from the depots to the customers such that the demands of the customers are met at the *least possible total cost*.

The transportation problem is a prototypical *sypply-and-demand* problem. For consistency we require: $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$.

The transportation problem can be modeled by a Linear Program, using indicator variables x_{ij} for the quantity to be transported from depot i to customer j:

minimize $\quad \sum c_{ij}x_{ij}$

subject to

$\sum_{j=1}^{n} x_{ij} \leq a_i \quad$ (for $i = 1, \ldots, m$) $\quad$ ('every depot has enough supply')

$\sum_{i=1}^{m} x_{ij} = b_j \quad$ (for $j = 1, \ldots, n$) $\quad$ ('every customer demand is met')

$x_{ij} \geq 0 \quad$ (for $1 \leq i \leq m$ and $1 \leq j \leq n$)

Alert readers will notice that the $\leq$-sign in the first constraint of the LP can actually be an $=$-sign: we can assume that *all* goods of the depots are actually 'transported', by (not) transporting everything that the customers do not need to a 'dump' at zero cost. Formally:

Proposition 9.1 *Without loss of generality we may assume that* $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$.

Proof: We assumed that $\sum_{i=1}^m a_i \geq \sum_{j=1}^n b_j$. If $\sum_{i=1}^m a_i > \sum_{j=1}^n b_j$, then invent a new customer $n+1$ with a demand $b_{n+1} = \sum_{i=1}^m a_i - \sum_{j=1}^n b_j$. Let $c_{i,n+1} = 0$ for $i = 1, \ldots, m$. Consider the transportation problem for this new instance. Of course we must include variables $x_{i,n+1}$ for $i = 1, \ldots, m$ now in the LP model.

The new model is equivalent to the old, as customer $n+1$ merely acts as a dump: transporting goods from any depot to $n+1$ costs nothing and thus is equivalent to leaving the goods that are not needed to satisfy the demands of customers $1, \ldots, n$ at the depots. In the new model we have $\sum_{i=1}^m a_i = \sum_{j=1}^{n+1} b_j$. ■

By 9.1 we may assume that $\sum_{j=1}^n x_{ij} = a_i$ in the general model of the transportation problem:

(TrP)

minimize $\sum c_{ij}x_{ij}$

subject to

$\sum_{j=1}^n x_{ij} = a_i \quad (i = 1, \ldots, m)$
$\sum_{i=1}^m x_{ij} = b_j \quad (j = 1, \ldots, n)$
$x_{ij} \geq 0 \; (1 \leq i \leq m, \, 1 \leq j \leq n)$.

Lemma 9.2 *Every instance of the transportation problem has a feasible solution.*

Proof: One way of seeing this is the following. Transport all goods from the first depot to the first customer and following, then continue with the goods of depot 2 and so on, stocking a customers with b_j goods before passing to the next customer. Because $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, this precisely satisfies all customer demands. The quantities x_{ij} transported from i to j give a feasible solution.

A different way is the following. Say $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j = H$. Choose $x_{ij} = \frac{a_i b_j}{H}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then

$$\sum_{j=1}^n x_{ij} = \sum_{j=1}^m \frac{a_i b_j}{H} = \frac{a_i}{H} \sum_{j=1}^m b_j = a_i \qquad \sum_{i=1}^m x_{ij} = \sum_{i=1}^m \frac{a_i b_j}{H} = \frac{b_j}{H} \sum_{i=1}^m a_i = b_j$$

showing that the constraints in the LP are fulfilled. Obviously $\frac{a_i b_j}{H} \geq 0$ if $a_i \geq 0$ and $b_j \geq 0$. Thus all constraints are met and this solution is feasible. ■

9.3 Modeling the transportation problem as a flow problem

In the previous lecture we saw that assignment problems can be reduced to solving *flow problems* on bipartite networks. For the transportation problem we will show a connection to a very useful general type of flow problem, namely flow with *costs*.

9.3.1 The minimum cost flow problem

Consider a (directed) flow network $G =< V, E, b, c, s, t >$ with:

> $b_{ij} \geq 0$, the capacity of edge ij
>
> $c_{ij} \geq 0$, the cost of transporting one unit of flow through ij
>
> a source s and a sink t with in-degree(s) = out-degree(t) = 0.

A feasible flow f in G must fulfill the following constraints:

- $0 \leq f(i, j) \leq b(i, j)$,
- $\sum_{(*,j)\in E} f(*, j) = \sum_{(j,*)\in E} f(j, *) \quad$ for all $j \in V - \{s, t\}$.

The second constraints are known as the *flow conservation* constraints. They imply that the amount of flow out of s is equal to the amount of flow into t.

Definition 9.3 (i) *The* value *of a flow f is* $v = \sum_{(s,*)\in E} f(s, *) = \sum_{(*,t)\in E} f(*, t) = |f|$.
(ii) *The* cost *of a flow f is* $||f|| = \sum_{(i,j)\in E} c(i, j)f(i, j)$

The *minimum cost flow problem* is defined as follows:

> Given a value $v_0 \geq 0$, determine a flow f from s to t such that $|f| = v_0$ with least possible cost $||f||$.

9.3.2 The equivalence theorem

The transportation problem can be rephrased into the minimum cost flow problem *and vice versa*. We show this in two steps.

Lemma 9.4 *Every instance of the transportation problem can be 'reduced' to a minimum cost flow problem.*

Proof: Consider an arbitrary instance of the transporting problem TrP as given above:

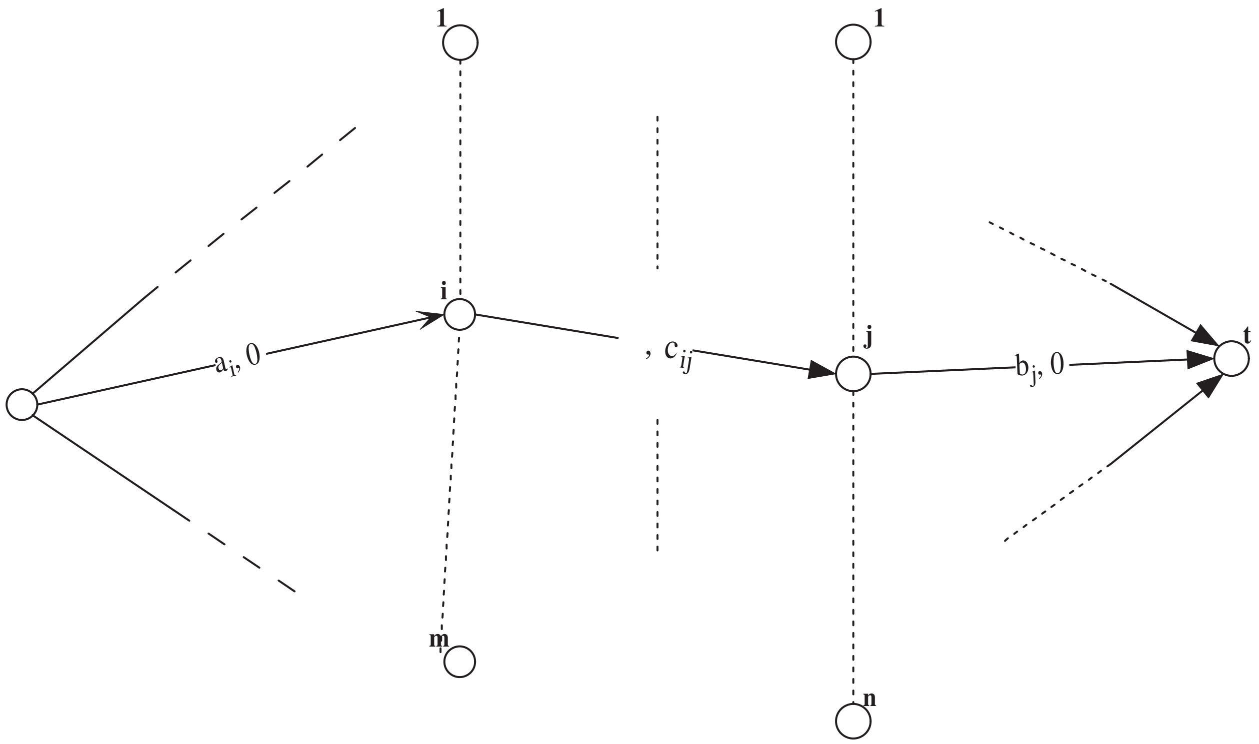
minimize $\quad \sum c_{ij} x_{ij}$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i \qquad \sum_{i=1}^{m} x_{ij} = b_j \qquad x_{ij} \ge 0.$$

Design a network G_{TrP} as follows. It consists of a source s, m nodes $(1, \ldots, m)$ corresponding to the depots, n nodes corresponding to the customers, and a sink t. G_{TrP} has the following edges:

- an edge from s to every depot i with capacity a_i and cost 0,
- an edge from every depot i to every customer j with capacity ∞ and cost c_{ij}, and
- an edge from every customer j to the sink t with capacity b_j and cost 0.



Let $v_{TrP} = \sum_{i=1}^{m} a_i \; (= \sum_{j=1}^{n} b_j)$. Consider the minimum cost flow problem in G_{TrP} for value v_{TrP}.

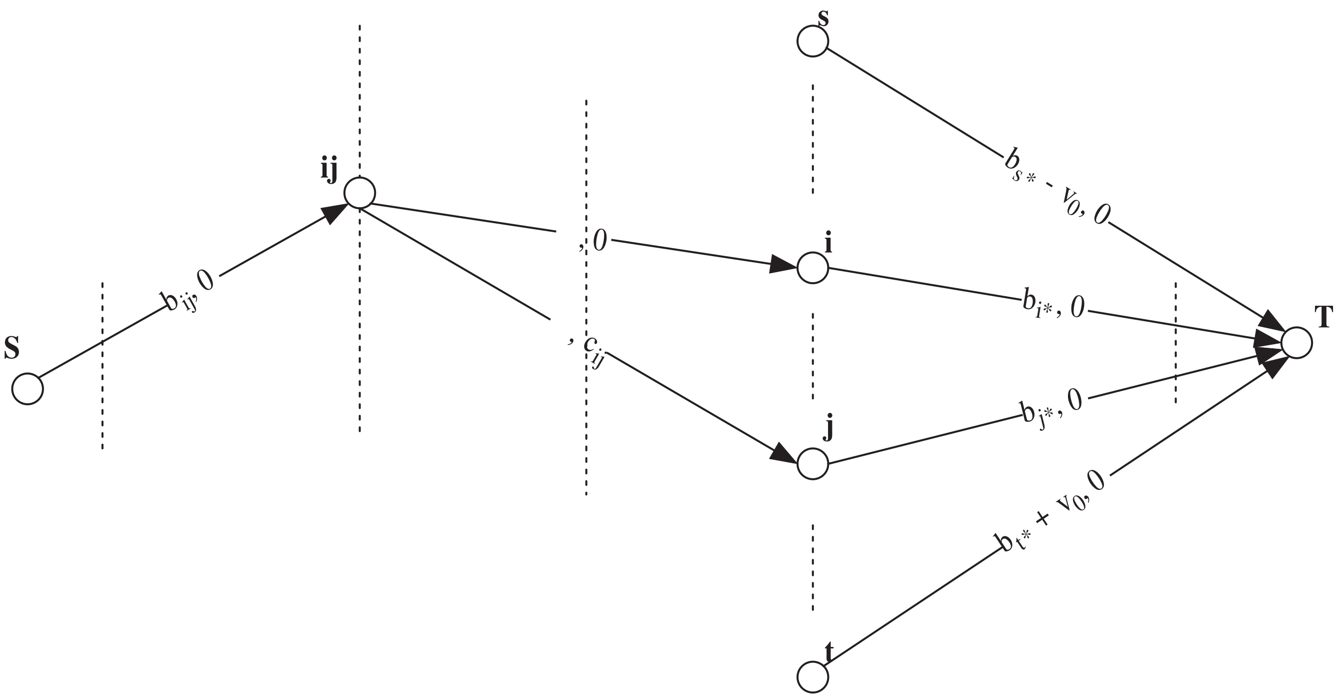
The required flow value forces the outgoing edges of s and the ingoing edges of t to be filled to their full capacity ('saturated'). By taking $x_{ij} = f(i, j)$ it is seen that the constraints in the TrP are precisely equal to the flow conservation constraints in the nodes i $(1 \le i \le m)$ and j $(1 \le j \le n)$ in G_{TrP}. The goal functions are equal as well. Thus every transportation problem can be reduced to this corresponding minimum cost flow problem. ■

The rather more surprising fact is the converse to this lemma.

Lemma 9.5 *Every instance of the minimum cost flow problem can be 'reduced' to a transportation problem.*

Proof: Consider an arbitrary instance of the minimum cost flow problem, based on a network $G =< V, E, b, c, s, t >$ and a certain desired flow value v_0. We will reduce the problem to a minimum cost flow problem in a network G_{TrP} of the type described in Lemma 9.3 that is directly equivalent to a transportation problem. G_{TrP} consists of: a new source S, nodes o^{ij} corresponding to the edges of the original network, n nodes (including s and t) corresponding to the nodes of the original network, and a new sink T. The edge-nodes act as depots, the nodes as customers. G_{TrP} has the following edges, where we will use the following notation: $b_{i*} = \sum_{ij \in E} b_{ij}$:

- an edge from S to every depot o^{ij} with capacity b_{ij} and cost 0,
- an edge from every depot o^{ij} to 'customer' i with capacity ∞ and cost 0,
- an edge from every depot o^{ij} to 'customer' j with capacity ∞ and cost c_{ij},
- an edge from every depot o^{ij} to customers $\neq i, j$ with capacity ∞ and cost ∞,
- an edge from customer s to sink T with capacity $b_{s*} - v_0$ and cost 0,
- an edge from every customer $j \neq s, t$ with capacity $b_{j,*}$ and cost 0,
- an edge from customer t to sink T with capacity $b_{t*} + v_0$ and cost 0.



G_{TrP} indeed represents a transportation problem. This follows since

$$v_{TrP} = \sum_{(i,j)\in E} b_{ij} = \sum_i b_{i*} = b_{s*} + \sum_{i\neq s,t} b_{i*} + b_{t*} = (b_{s*} - v_0) + \sum_{i\neq s,t} b_{i*} + (b_{t*} + v_0).$$

and thus *the supply at* $S =$ *the demand at* T. Note also that $b_{s*} - v_0 \geq 0$, otherwise we cannot have a feasible flow in G. Thus the graph is of the correct form.

Claim 9.6 *Feasible flows of value* v_0 *in* G *correspond to feasible flows of* v_{TrP} *in* G_{TrP} *with the same cost value, and vice versa.*

Proof: ($\Rightarrow$) Suppose f is a feasible flow in G, where $|f| = v_0$ and certain cost $||f||$. Now design a feasible flow in G_{TrP} as follows:

send b_{ij} flow from S to o^{ij}

divide the flow from o^{ij} in such way that:

- $f(i,j)$ flow goes to j
- $b_{ij} - f(i,j)$ flow is diverted to i

saturate the edges towards T.

The flow conservation constraints are obviously satisfied in all nodes o^{ij}. Let us now look at the flow that enters nodes $s,\ldots,i,\ldots,j,\ldots,t$. For every such node i the incoming flow is equal to the *diverted* flow from nodes o^{i*} plus the *direct* flow from nodes o^{*i}:

$$\sum_{(i,j)\in E} (b_{ij} - f(i,j)) + \sum_{(j,i)\in E} f(j,i) = b_{i*} + (\sum_{(*,i)\in E} f(*,i) - \sum_{(i,*)\in E} f(i,*)).$$

This equals:

- b_{i*} for $i \neq s,t$ (use the flow-constraints),
- $b_{s*} - v_0$ for $i = s$, and
- $b_{t*} + v_0$ for $i = t$.

Send this flow through to node T and we have the defined the whole flow. Note that the costs of the flows in both networks are equal, namely $\sum_{(i,j)\in E} c_{ij} f(i,j)$.

($\Leftarrow$) Conversely, consider a feasible flow of value $v_{TrP}i$ in G_{TrP}. Suppose the amount of flow that goes through the edge $o^{ij} \to j$ is x_{ij}. We claim: if we take $f(i,j) = x_{ij}$ in the original graph we have a feasible flow with $|f| = v_0$ and same cost. To see this,we argue as follows.

If x_{ij} flow goes to node j, then $b_{ij} - x_{ij}$ flow goes to node i in G_{TrP}. Thus we may conclude that $0 \leq x_{ij} \leq b_{ij}$ for all $(i,j) \in E$ (first flow constraint). If we look at the flow conservation constraints in a node $i \neq s,t$ in G_{TrP} we find:

$$b_{i*} = \sum_{(*,i)\in E} x_{*i} + \sum_{(i,*)\in E} (b_{i*} - x_{i*})a = b_{i*} + (\sum_{(*,i)\in E} x_{*i} - \sum_{(i,*)\in E} x_{i*}).$$

So, $\sum_{(*,i)\in E} x_{*i} = \sum_{(i,*)\in E} x_{i*}$ (flow conservation), therefore the flow is feasible. Moreover, in the same manner one can verify that for $i = s$ and $i = t$ the flow has value v_0. Finally, in both cases the costs are $\sum c_{ij} x_{ij}$. ■

We have thus found a correct way to transform a minimum cost flow problem into a 'minimum cost' transportation problem. ■

The two lemmas combined give our main result.

Theorem 9.7 (Wagner, 1959) *The minimum cost flow problem is* equivalent *to the transportation problem.*

The result implies that efficient algorithms to solve the minimum cost flow problem can be converted into efficient algorithms to solve the transportation problem *and vice versa.*

9.4 The assignment problem

We recall the problem of matching n persons to n tasks (perfect bipartite matching). Suppose that every person can do all tasks but at certain *costs.* Let $c_{ij} \geq 0$ be the cost charged by person i for performing task j. The *assignment problem* asks for a perfect matching of least possible total cost.

Consider the following LP-model of the assignment problem:

minimize $\sum c_{ij} x_{ij}$

subject to

$$\begin{array}{ll} \sum_{j=1}^{n} x_{ij} = 1 & (\text{for } i = 1, \ldots, n) \\ \sum_{i=1}^{n} x_{ij} = 1 & (\text{for } j = 1, \ldots, n) \\ x_{ij} \geq 0 & (\text{for } 1 \leq i, j \leq n) \\ x_{ij} \in \{0, 1\} & (**) \end{array}$$

It follows that the assignment problem is a special case of the transportation problem with $a_i = 1$ and $b_j = 1$ for all $1 \leq i, j \leq n$ and with the $0 - 1$ constraint added for the x_{ij}. We note a special property of the transportation problem (not proved in class).

Fact 9.8 *Suppose all a_i and b_j are integer in the transportation problem. Then the coefficients of all basic feasible solutions of the problem are integral.*

Lemma 9.9 *Constraint (**) in the assignment problem can be omitted without changing the optimal solution to the problem.*

Proof: Without condition (**) we have a special case of the transportation problem: take $m = n$ and $a_i = b_i = 1$ for $1 \leq i, j \leq n$. By 9.8 we have that all basic feasible solutions of this problem are integral. Since the optimum of an LP is achieved in one of the basic feasible solutions, we know the optimum solution is integral. The constraints of the problem force the values to be $\in \{0, 1\}$. Thus condition (**) is automatically fulfilled and may be omitted. ■

9.4.1 Modeling the assignment problem as a shortest path problem

Consider a network G with weighted edges and no cycles of weight < 0. The shortest path problem in G is well-defined. We now show that there is a rather direct connection between the assignment problem and the single-pair shortest path problem.

Lemma 9.10 *The single-pair shortest path problem is 'reducible' to the assignment problem.*

Proof: Consider the graph G of a shortest path problem and label the nodes in G in such way that the source is 1 and the target is n. Let c_{ij} be the weight of edge $i \to j$ and take $c_{ij} = \infty$ for all $(i, j) \notin E$ and $c_{ii} = 0$ for all i. Consider the following assignment problem with variables $x_{12}, \cdots, x_{1n}, x_{22}, \cdots, x_{2n}, \cdots\cdots, x_{(n-1)2}, \cdots, x_{(n-1)n}$ in LP formulation:

minimize $\quad z = \sum_{i=1}^{n-1} \sum_{j=2}^{n} c_{ij}x_{ij}$

subject to

$\sum_{j=2}^{n} x_{ij} = 1 \qquad (\text{for } 1 \leq i \leq n-1)$

$\sum_{i=1}^{n-1} x_{ij} = 1 \qquad (\text{for } 1 \leq j \leq n)$

$x_{ij} \geq 0 \qquad (\text{for } 1 \leq i \leq n-1 \text{ and } 2 \leq j \leq n)$

By 9.8 this problem has an optimal solution which is integral. By the constraints of this problem the values of this optimum solution are further forced to be $\in \{0, 1\}$. One can imagine this solution to be a matrix (x_{ij}) with rows 1 to $n-1$ and columns 2 to n and all entries $\in \{0, 1\}$. Then the constraints force that *in each row and each column there is exactly one* 1 *and the other values are* 0.

Suppose that the shortest path from 1 to n has weight W and the optimum of the assignment problem is z^*.

Claim 9.11 $z^* = W$

Proof: ($\Rightarrow$) First we prove $z^* \leq W$. This follows because there is a feasible solution with weight W. Suppose $1, j_1, j_2, \ldots, j_r, n$ is a path of length (weight) W. Then take the solution with $x_{1j_1} = x_{j_1j_2} = \ldots = x_{j_rn} = 1$ and $x_{ii} = 1$ for all $i \neq j_1, \ldots, j_r$. For this feasible solution the goal function has exactly value W.

($\Leftarrow$) Now we prove $z^* \geq W$. Consider an optimal solution to the assignment problem as defined above. Now construct a 'chain' starting with the unique j_1 with $x_{1j_1} = 1$. The next link is determined as follows. Suppose we reached j_p and $j_p \neq n$. Determine j_{p+1} as the unique *column* in the matrix (x_{ij}) that has $x_{j_pj_{p+1}} = 1$. Now look at the chain $1, j_1, j_2, \ldots, j_p$. A new j_{p+1} can never be equal to an *earlier* j_i, since this would mean that we would have two entries with a 1 in the same column and this contradicts the constraints. So, the chain ends with n: $1, j_1, j_2, \ldots, n$ and can be as a path from 1 to n (and every c-value in it is $< \infty$ because we have a finite optimum). Now look at the indices $i \notin \{j_1, \ldots, j_r\}$. By the same argument one can see that the x_{ij}'s with value 1 divide into (directed) cycles in G. Since we assumed there are no negative cycles, these cycles all have weight ≥ 0. So it follows that the optimum $z^* \geq W + 0 + \cdots + 0 = W$.

So $z^* = W$ as was to be shown. ■

Thus we have transformed the shortest path problem to an assignment problem with the same optimum in an easy manner. ■

Lemma 9.12 *The assignment problem is 'reducible' to the single-pair shortest path problem.*

Proof: Consider an assignment problem in its regular LP form:

$$\begin{aligned}
&\text{minimize} \quad z = \sum c_{ij}x_{ij} \\
&\text{subject to} \\
&\qquad \textstyle\sum_{j=1}^{n} x_{ij} = 1 \qquad (\text{for } i = 1, \ldots, n) \\
&\qquad \textstyle\sum_{i=1}^{n} x_{ij} = 1 \qquad (\text{for } j = 1, \ldots, n) \\
&\qquad x_{ij} \geq 0 \qquad (\text{for } 1 \leq i, j \leq n)
\end{aligned}$$

Claim 9.13 *Without loss of generality we can assume $c_{ij} \geq 0$ for all $1 \leq i, j \leq n$.*

Proof: Suppose not all c_{ij} are ≥ 0. Take a C large enough so $C + c_{ij} \geq 0$ for all i and j. The assignment problem is clearly equivalent to the modified problem with goal function

$$z + C \cdot n = \sum c_{ij}x_{ij} + C \cdot n = \sum c_{ij}x_{ij} + C \cdot \sum x_{ij} = \sum (C + c_{ij}) \cdot x_{ij}$$

which clearly has all costs non-negative. ■

Now *rewrite* the problem by 'moving' the j-index 1 up:

$$\begin{aligned}
&\text{minimize} \quad z = \textstyle\sum_{i=1}^{n} \sum_{j=2}^{n+1} c_{ij}x_{ij} \\
&\text{subject to} \\
&\qquad \textstyle\sum_{j=2}^{n+1} x_{ij} = 1 \qquad (\text{for } i = 1, \ldots, n) \\
&\qquad \textstyle\sum_{i=1}^{n} x_{ij} = 1 \qquad (\text{for } j = 2, \ldots, n+1) \\
&\qquad x_{ij} \geq 0 \qquad (\text{for } 1 \leq i \leq n \text{ and } 2 \leq j \leq n+1)
\end{aligned}$$

Observe that this assignment problem has the same form as derived in Lemma 9.10 and we can in fact assume it comes from a graph G with weights c_{ij} as in the very same construction. Because the c_{ij} are ≥ 0, G indeed has no cycles of negative weight and its shortest path problem is well-defined. It easily follows that the optimum solution to the assignment problem equals the weight of the shortest path from 1 to $n + 1$ in G. ■

By the two lemmas we conclude the following quite remarkable result:

Theorem 9.14 *The assignment problem is* equivalent *to the single-pair shortest path problem.*

As the reductions are quite simple, the following conclusion follows as a special consequence.

Corollary 9.15 *The assignment problem can be solved* efficiently *with a shortest path algorithm in polynomial time.*

References

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