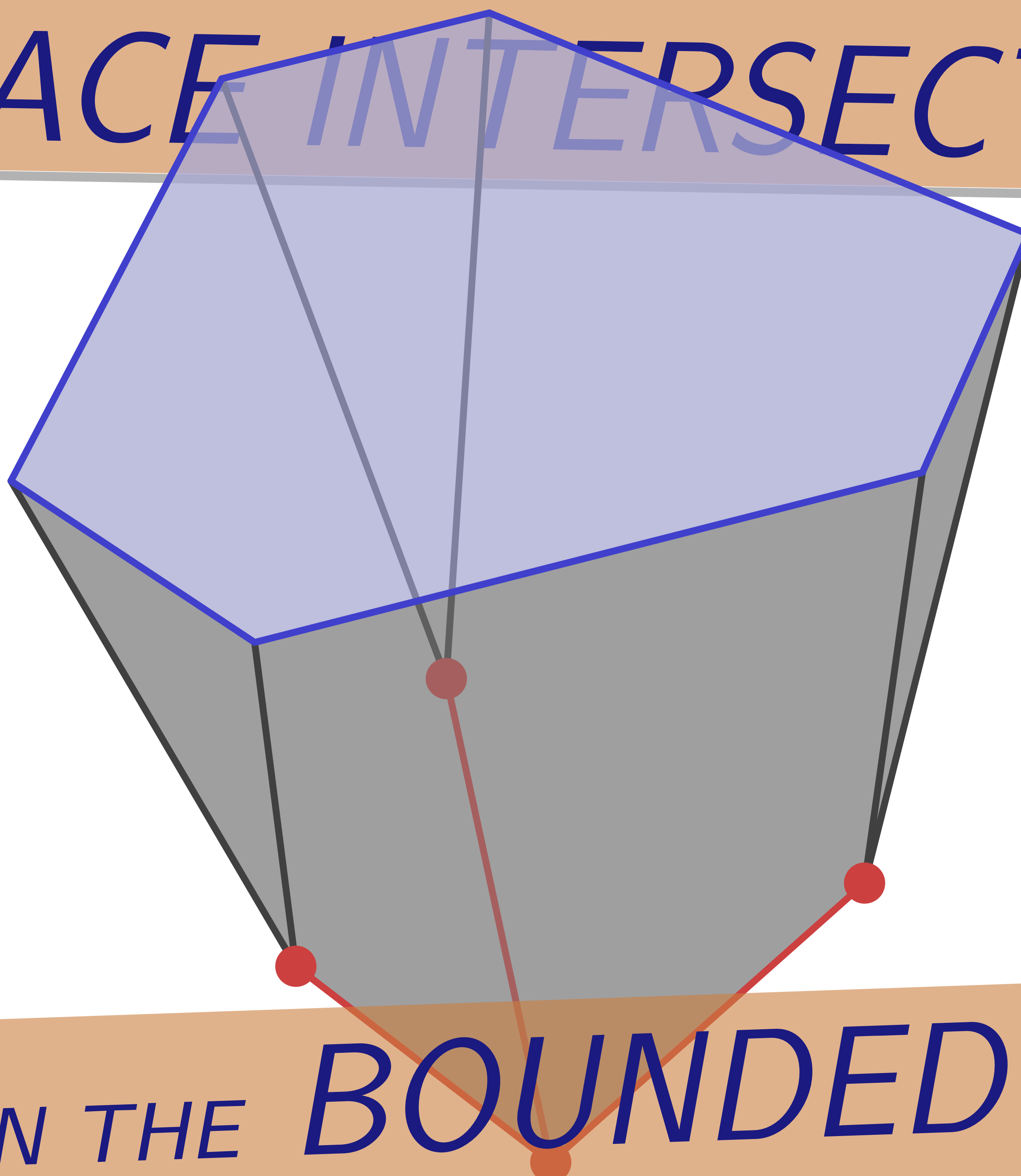
BOUNDS ON THE COMPLEXITY OF HALFSPACE INTERSECTIONS

David Eppstein

Maarten Löffler



WHEN THE BOUNDED FACES HAVE SMALL DIMENSION

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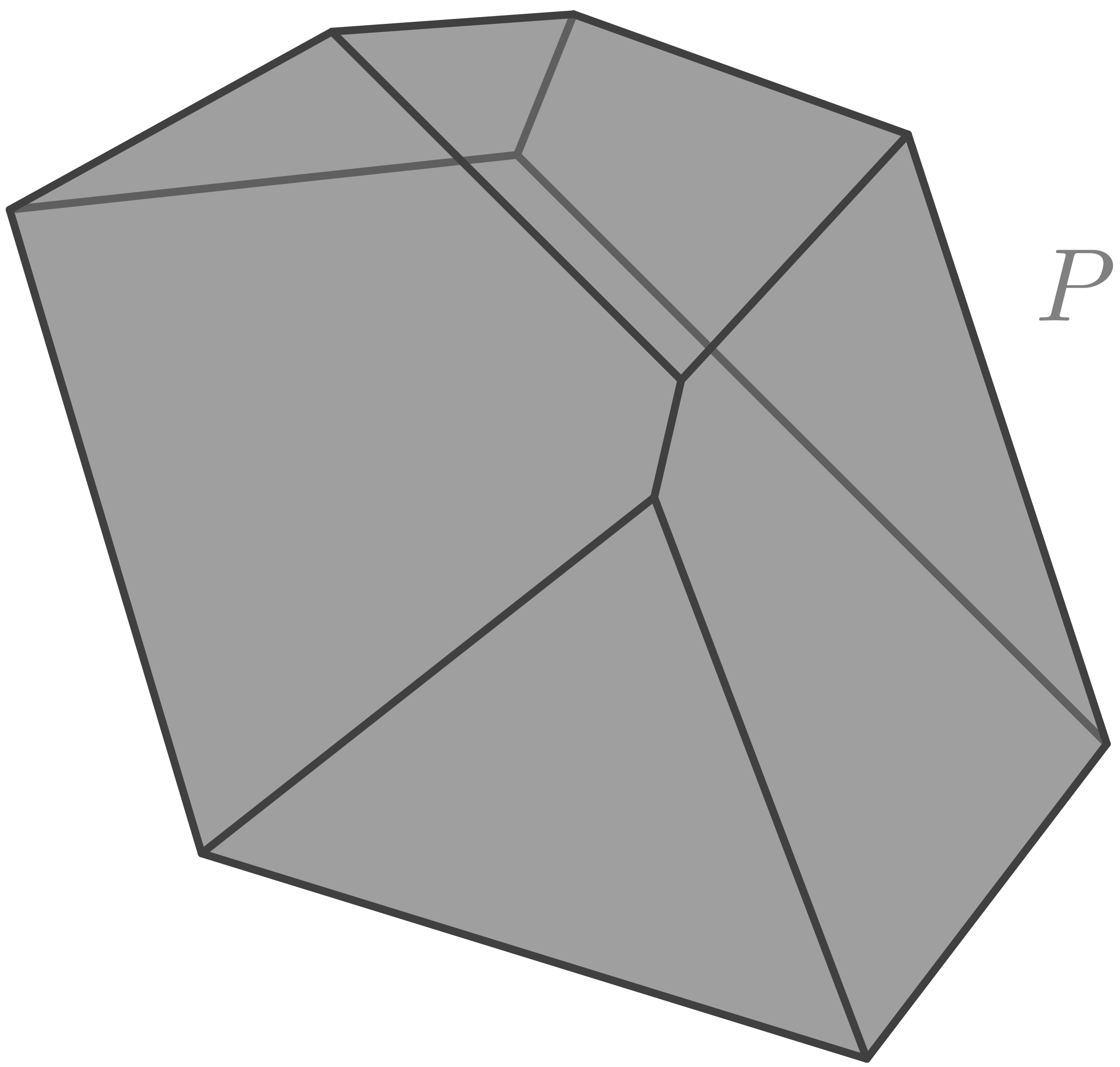

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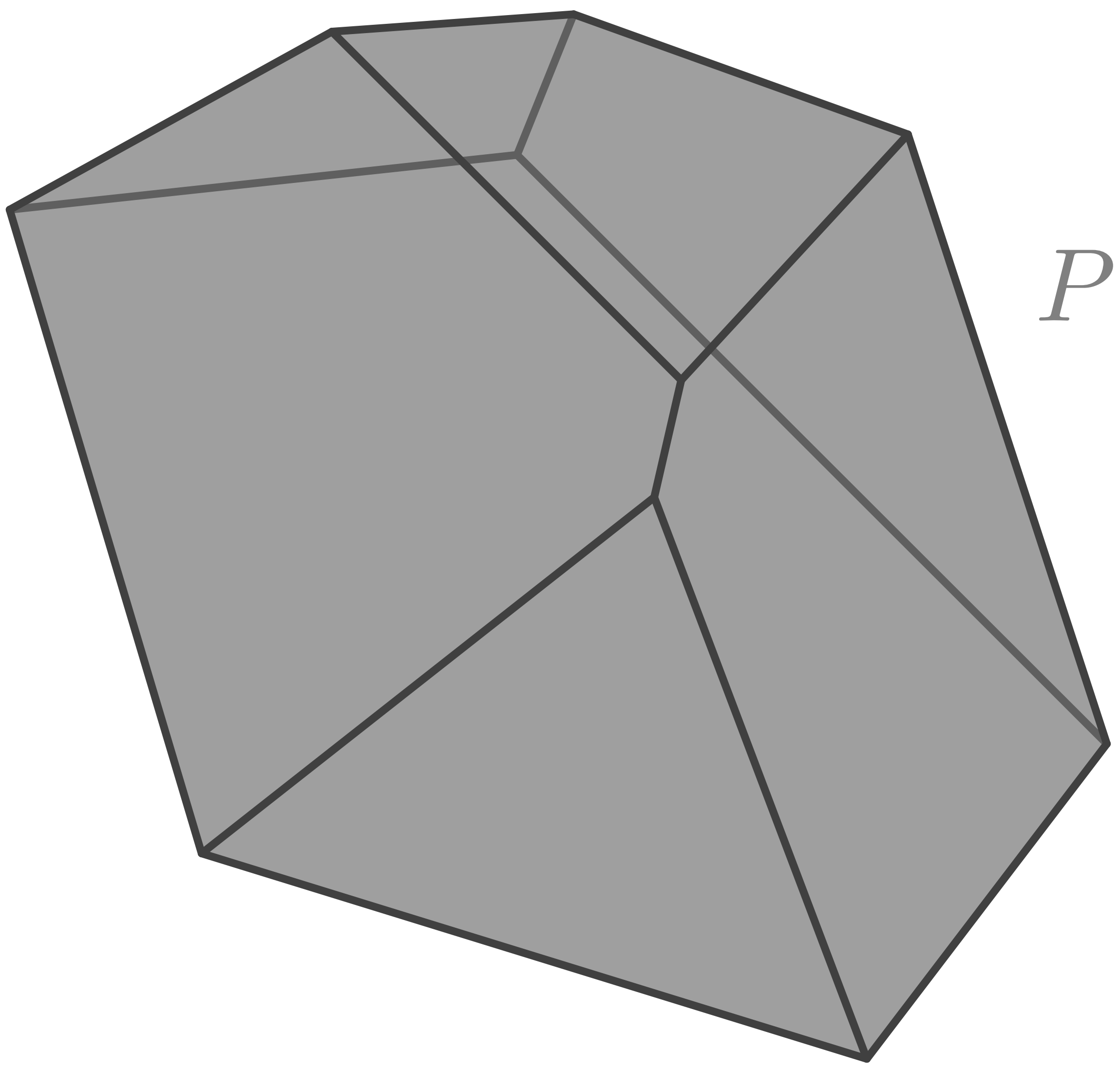

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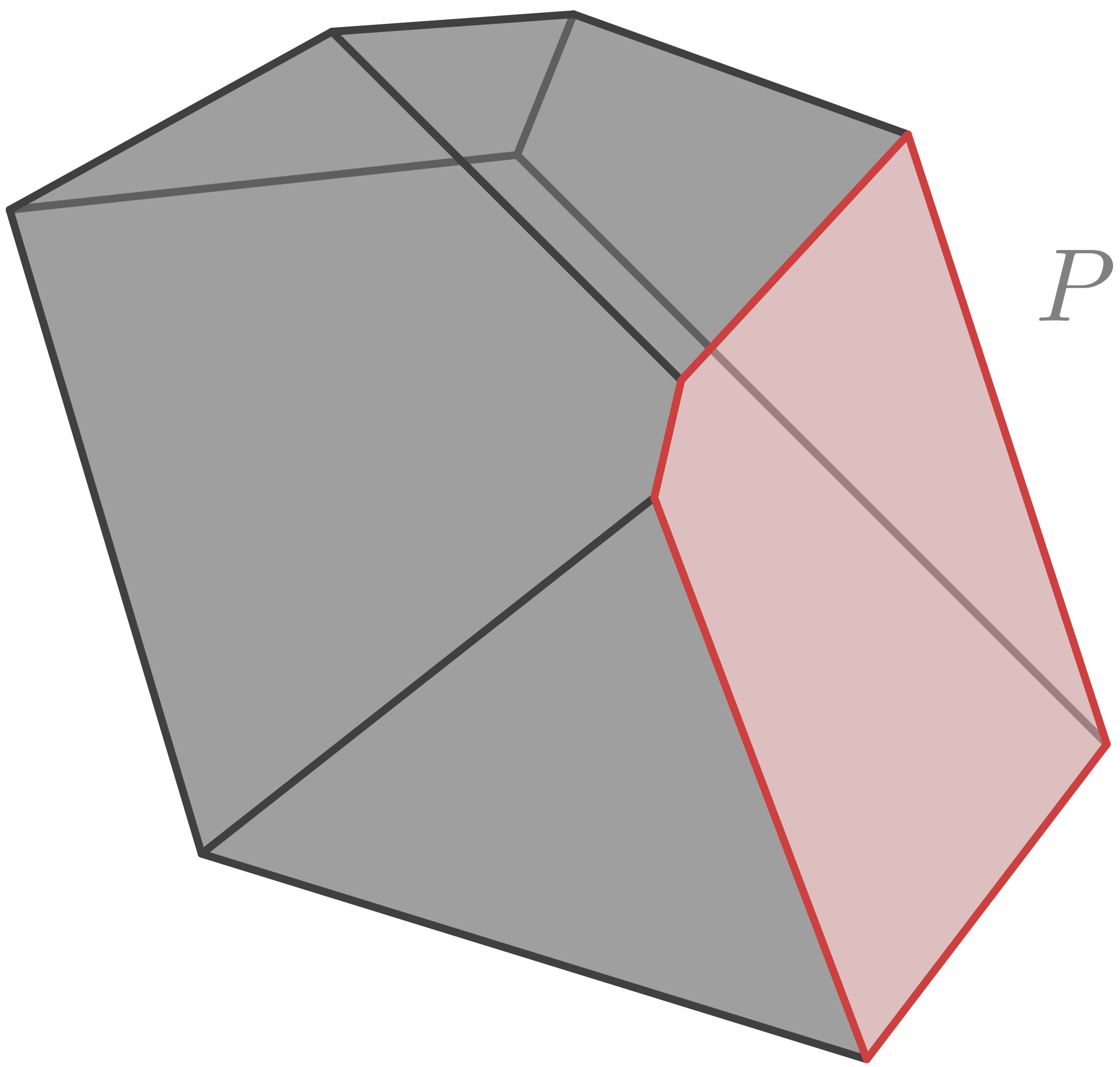

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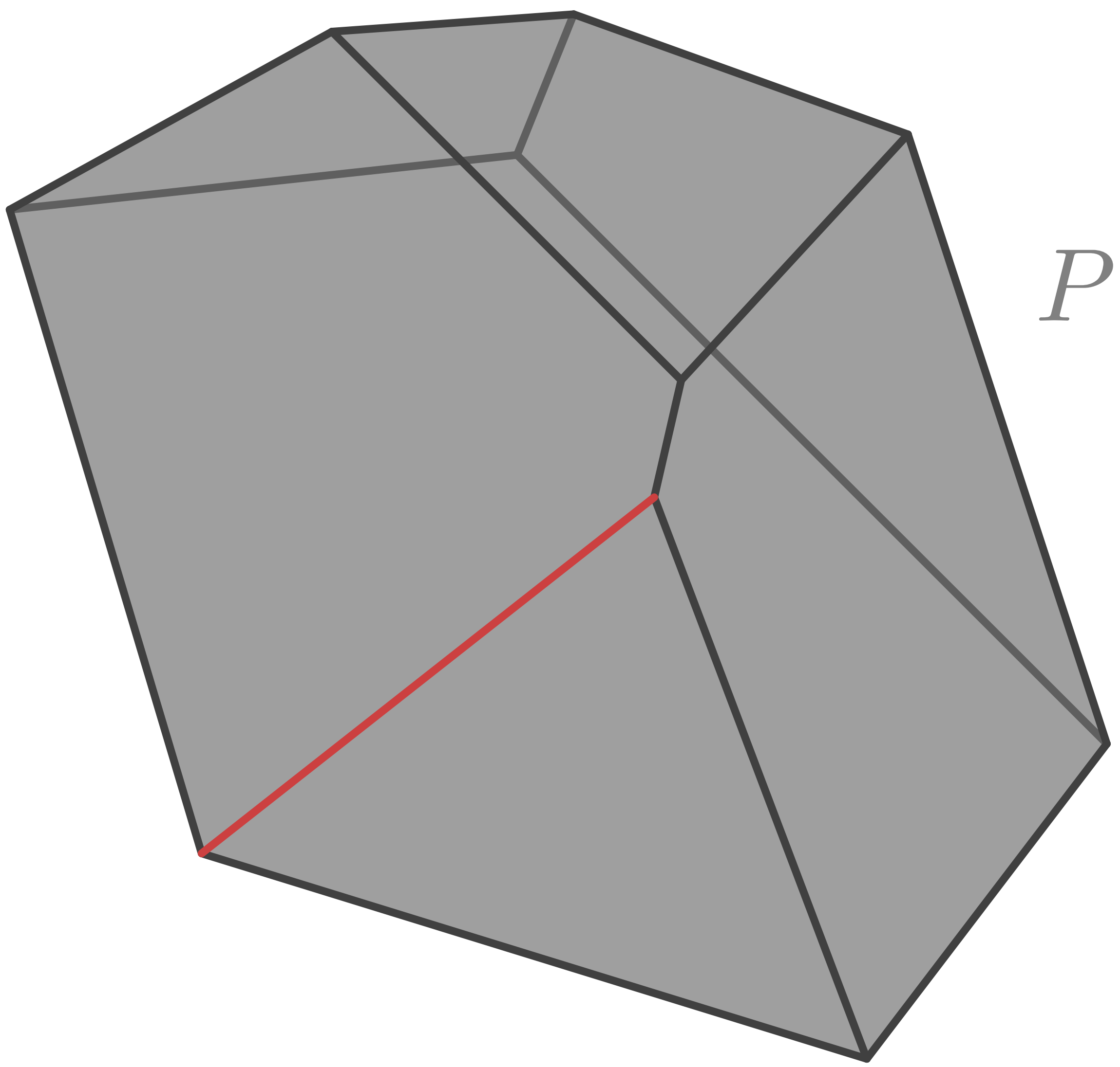

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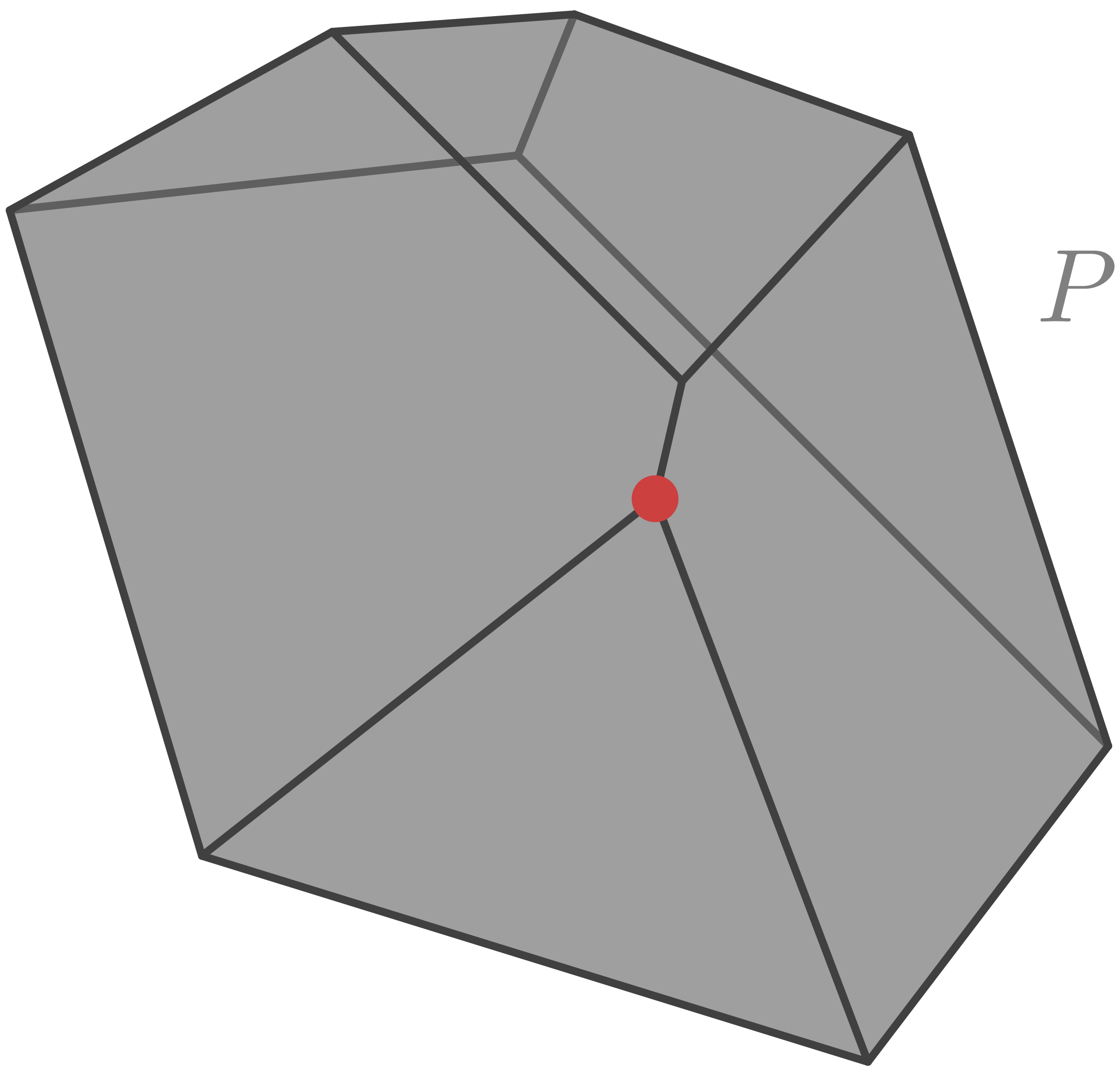

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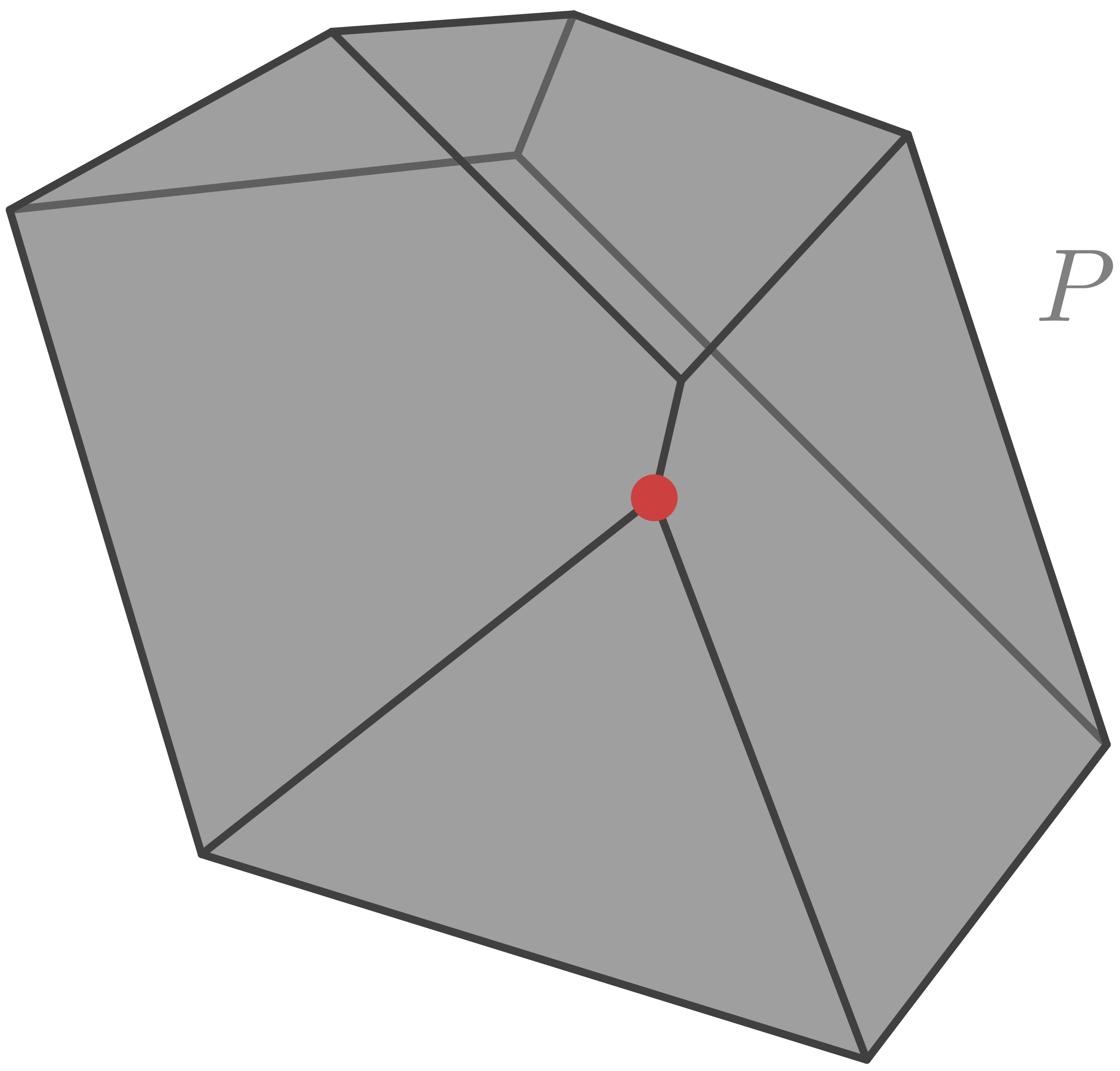

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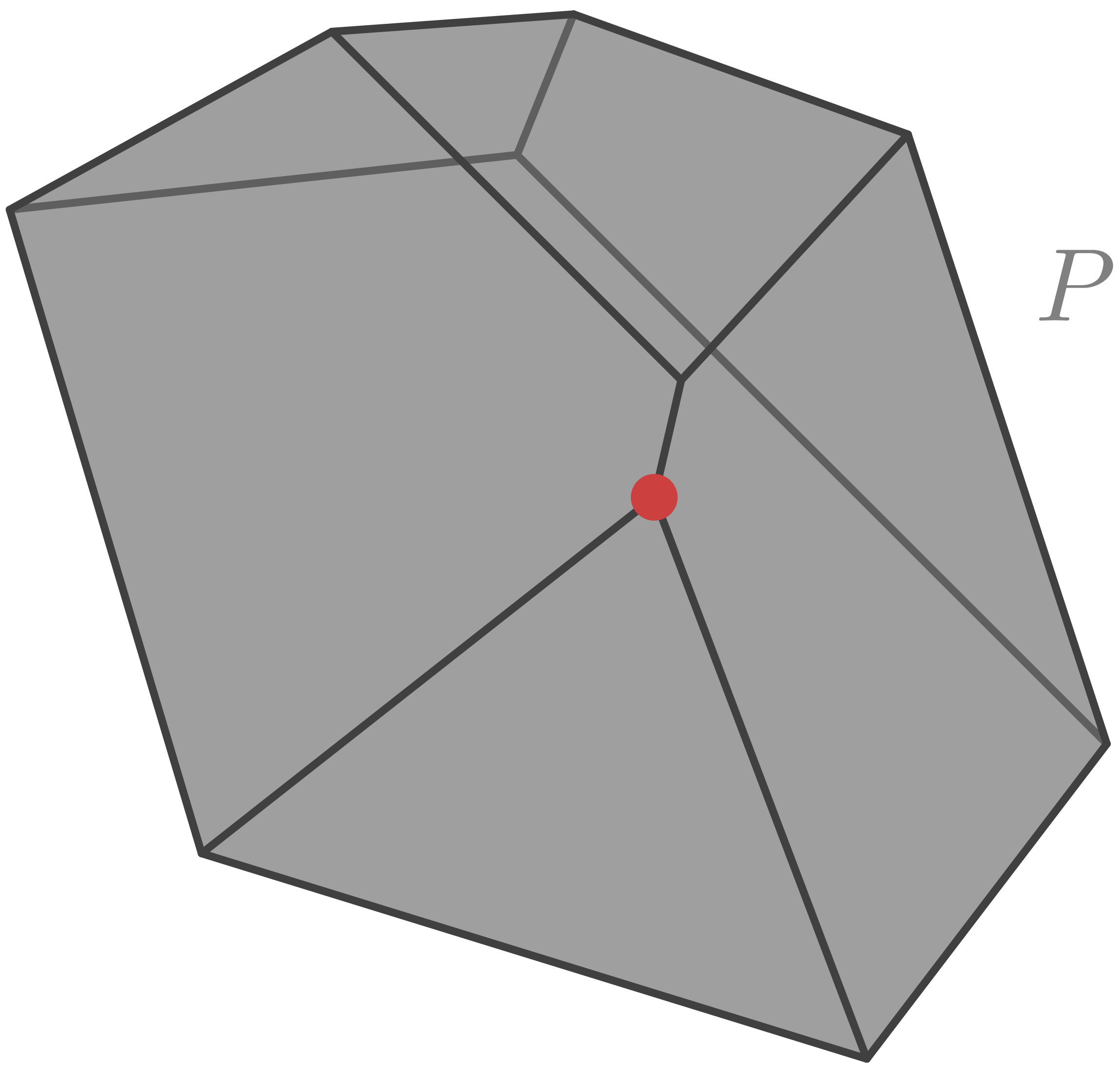

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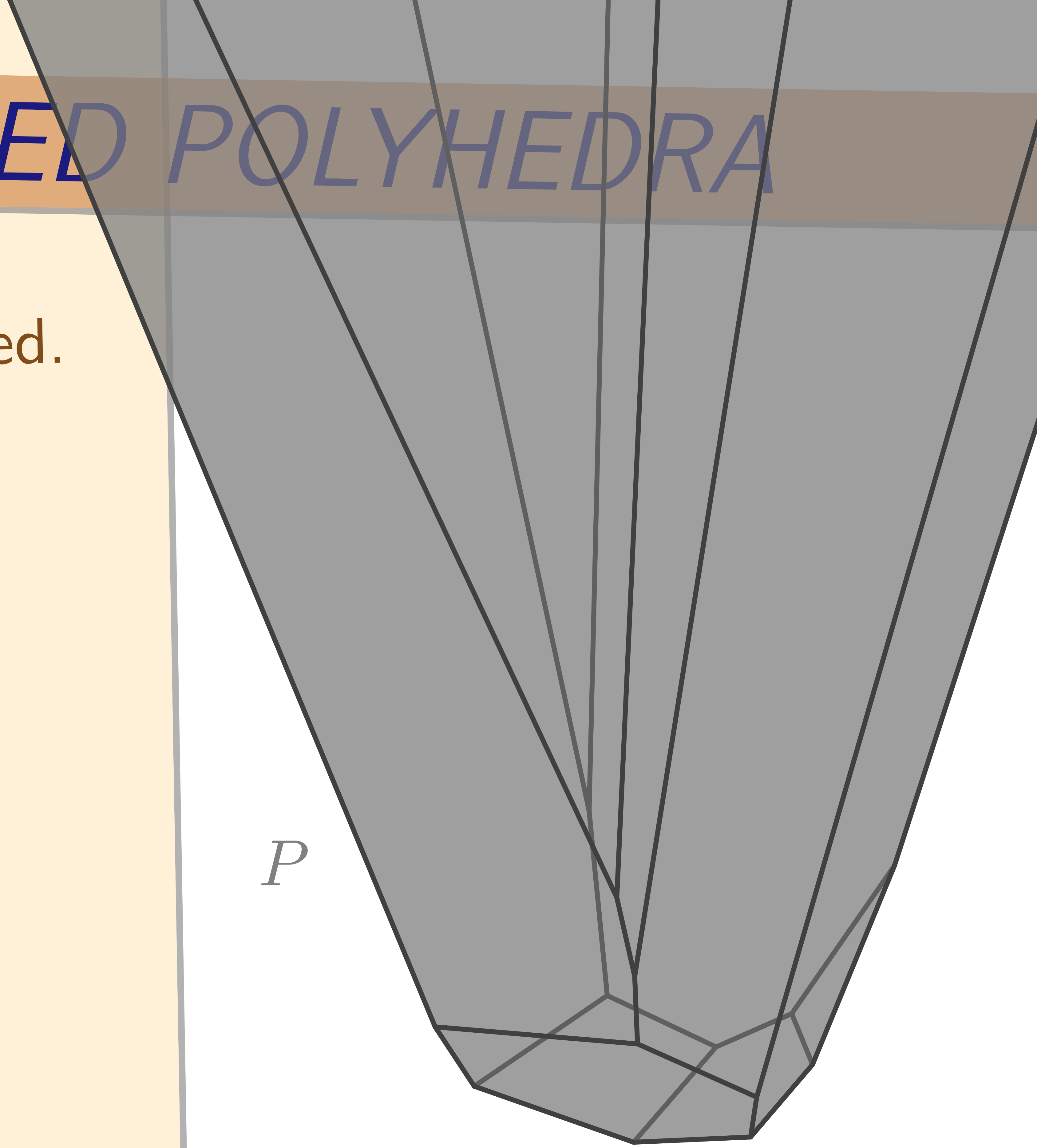

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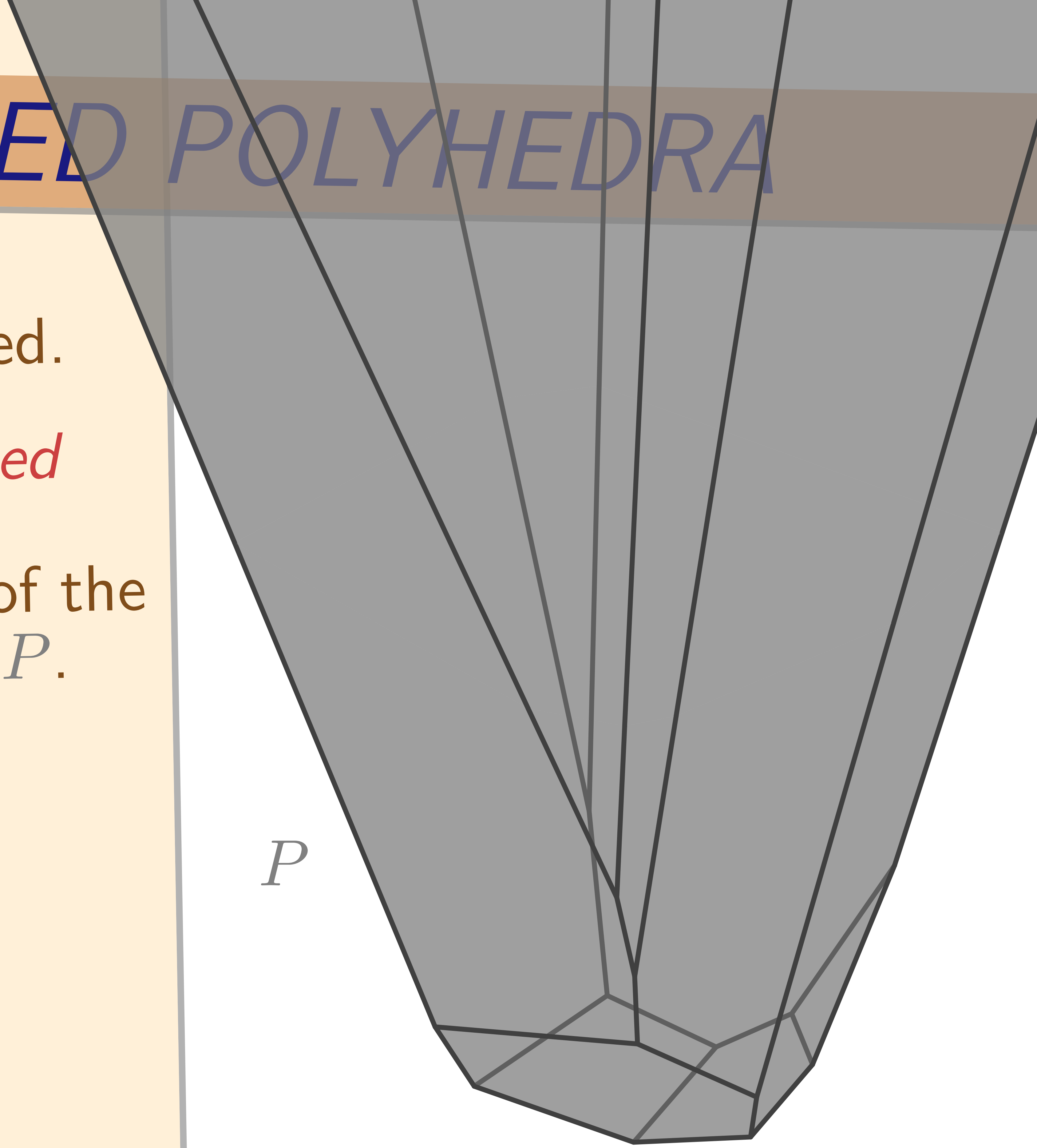
P_B is the subset of the bounded faces of P.



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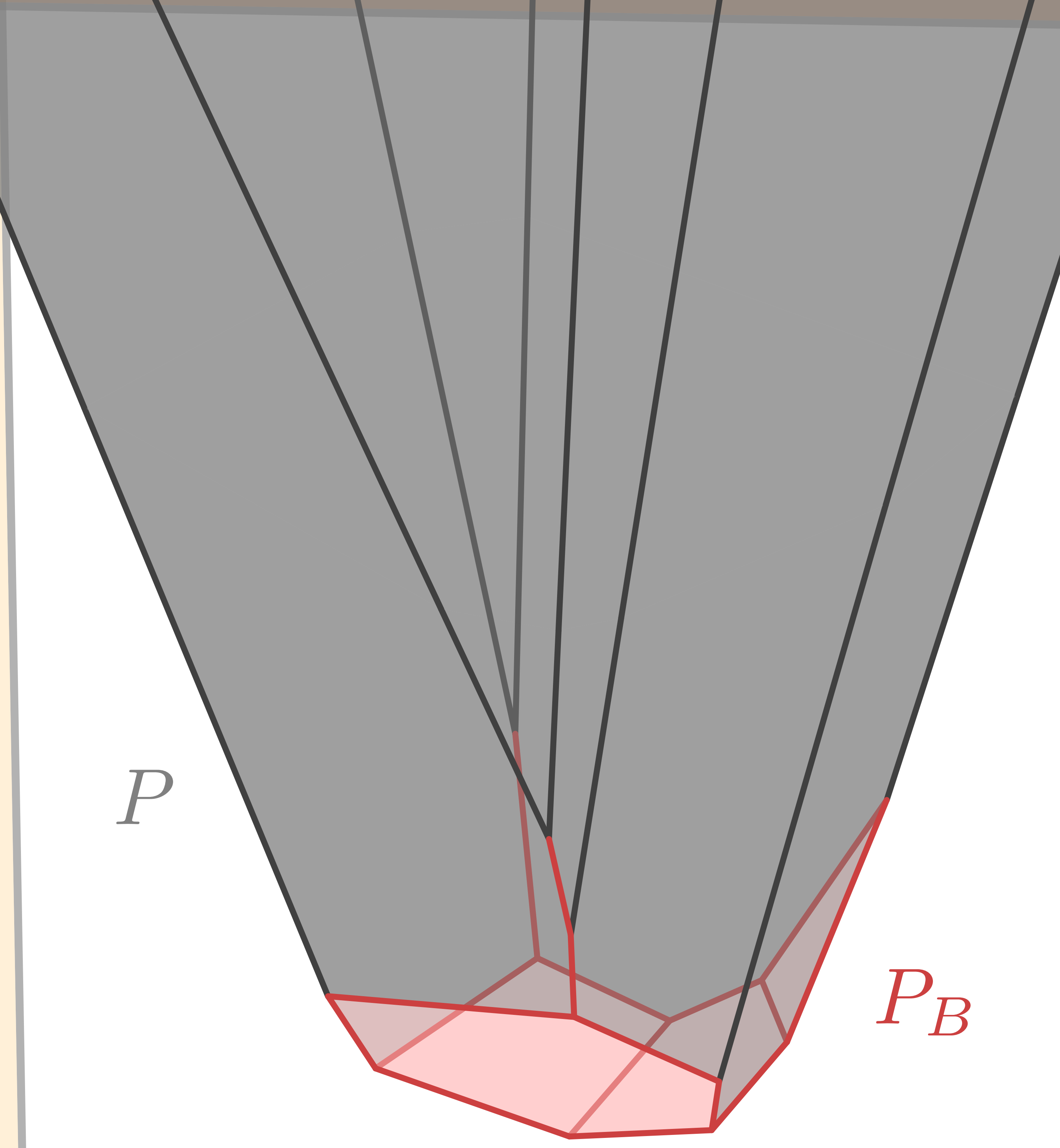

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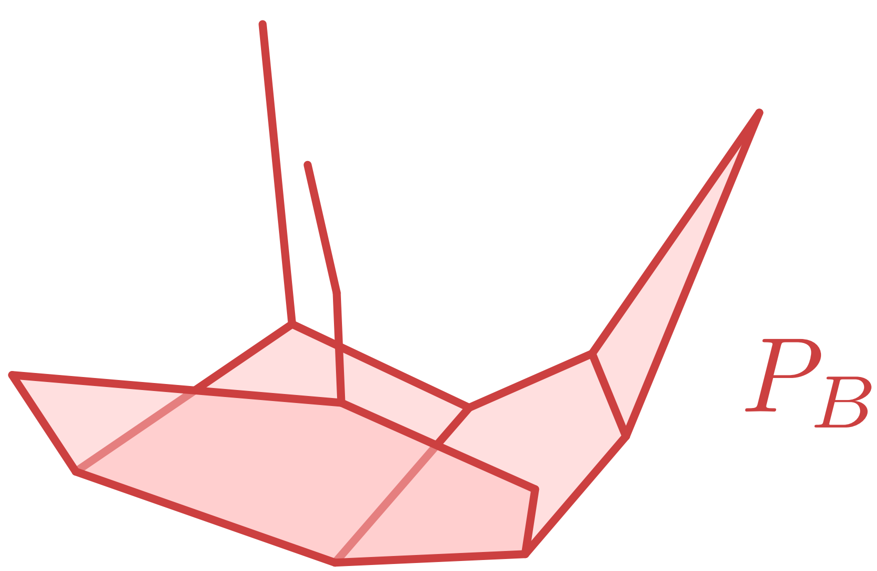

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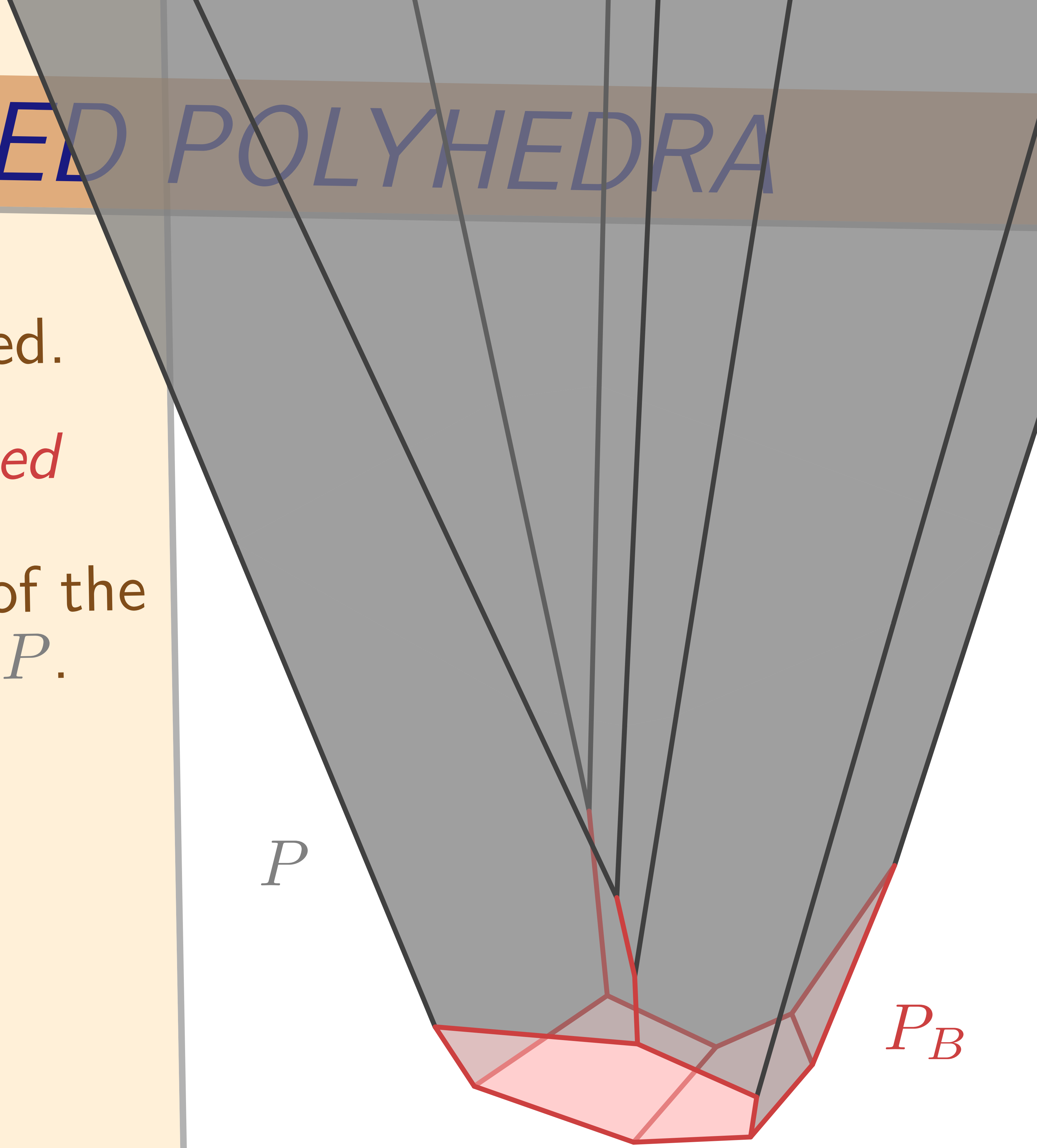

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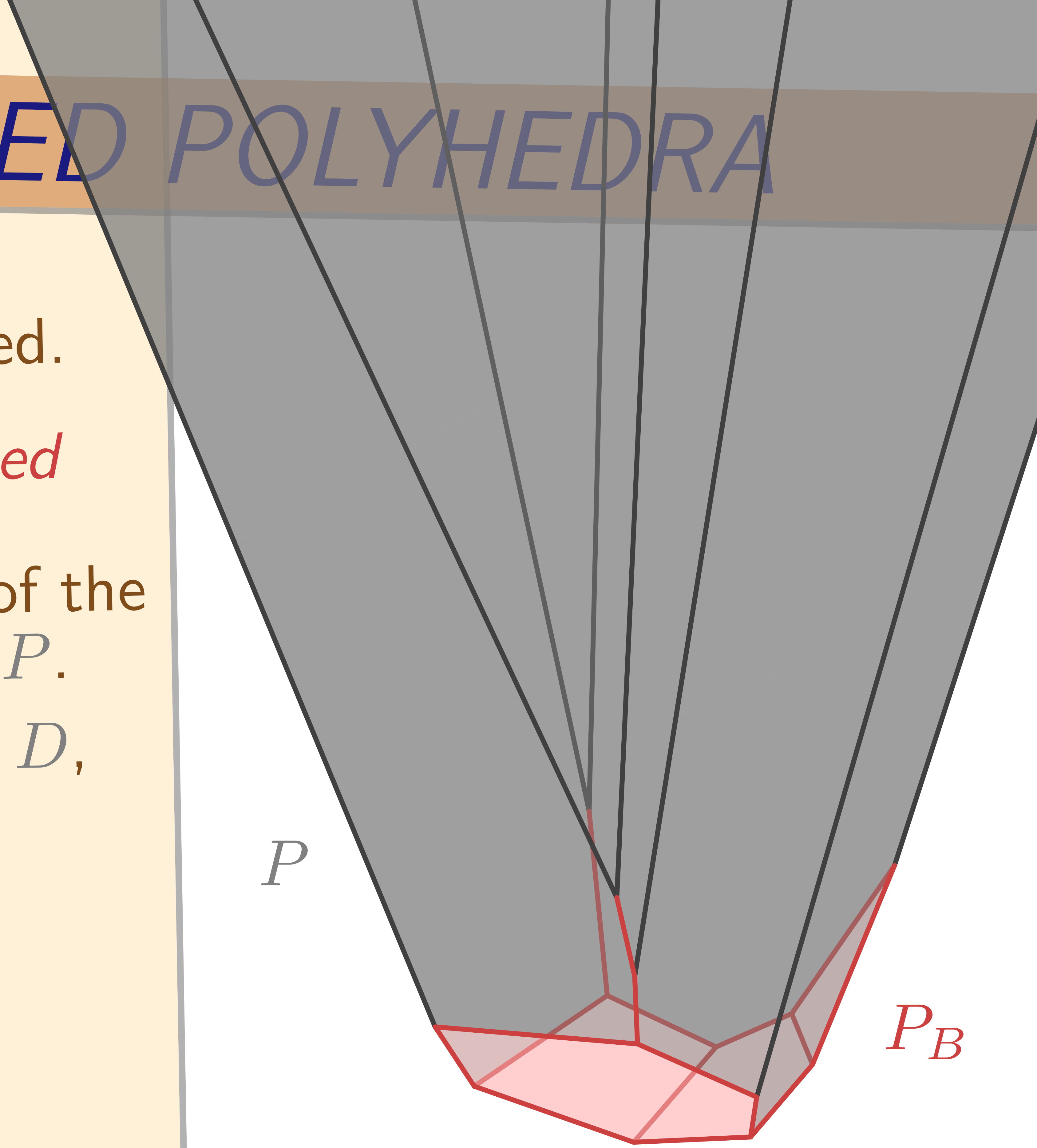

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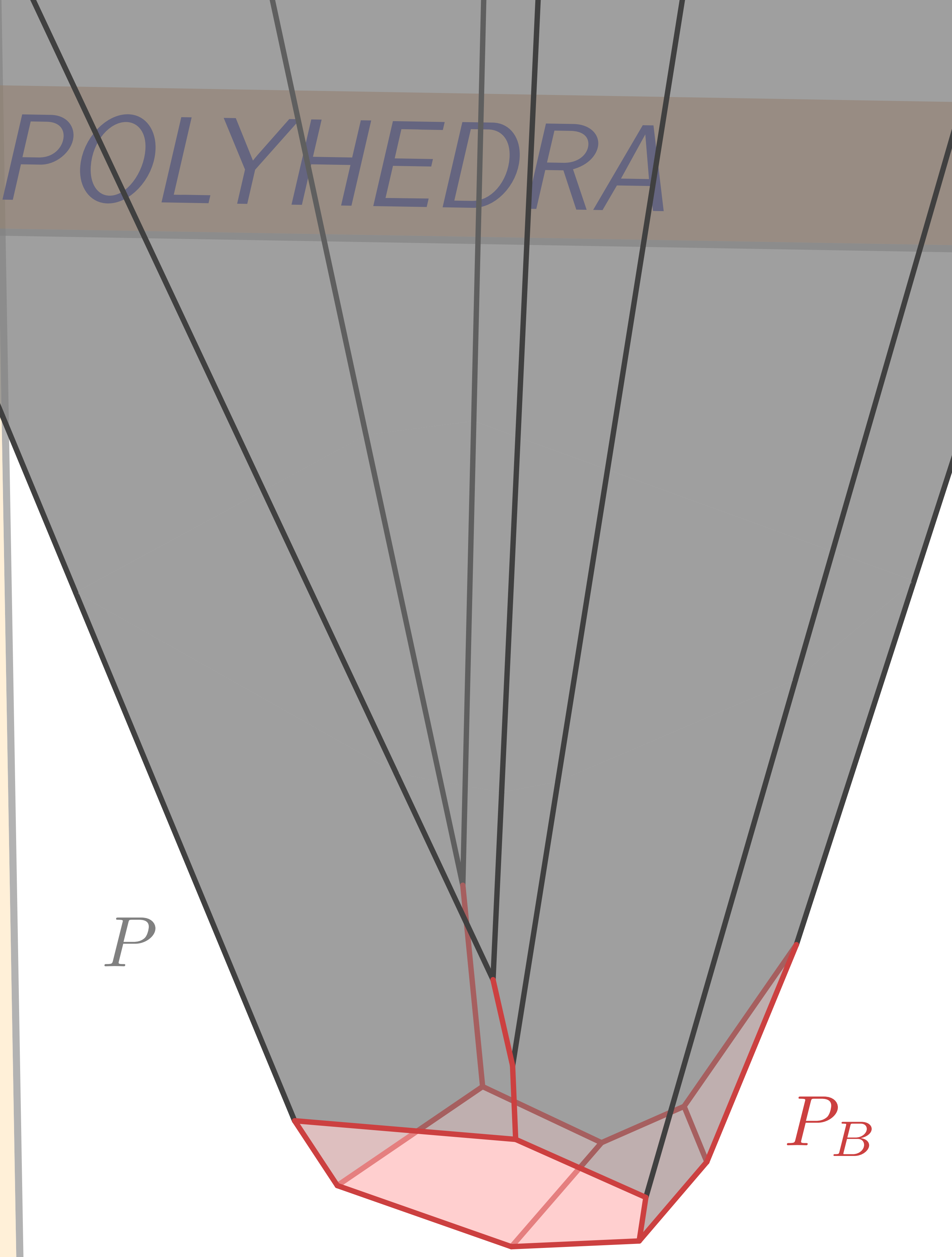

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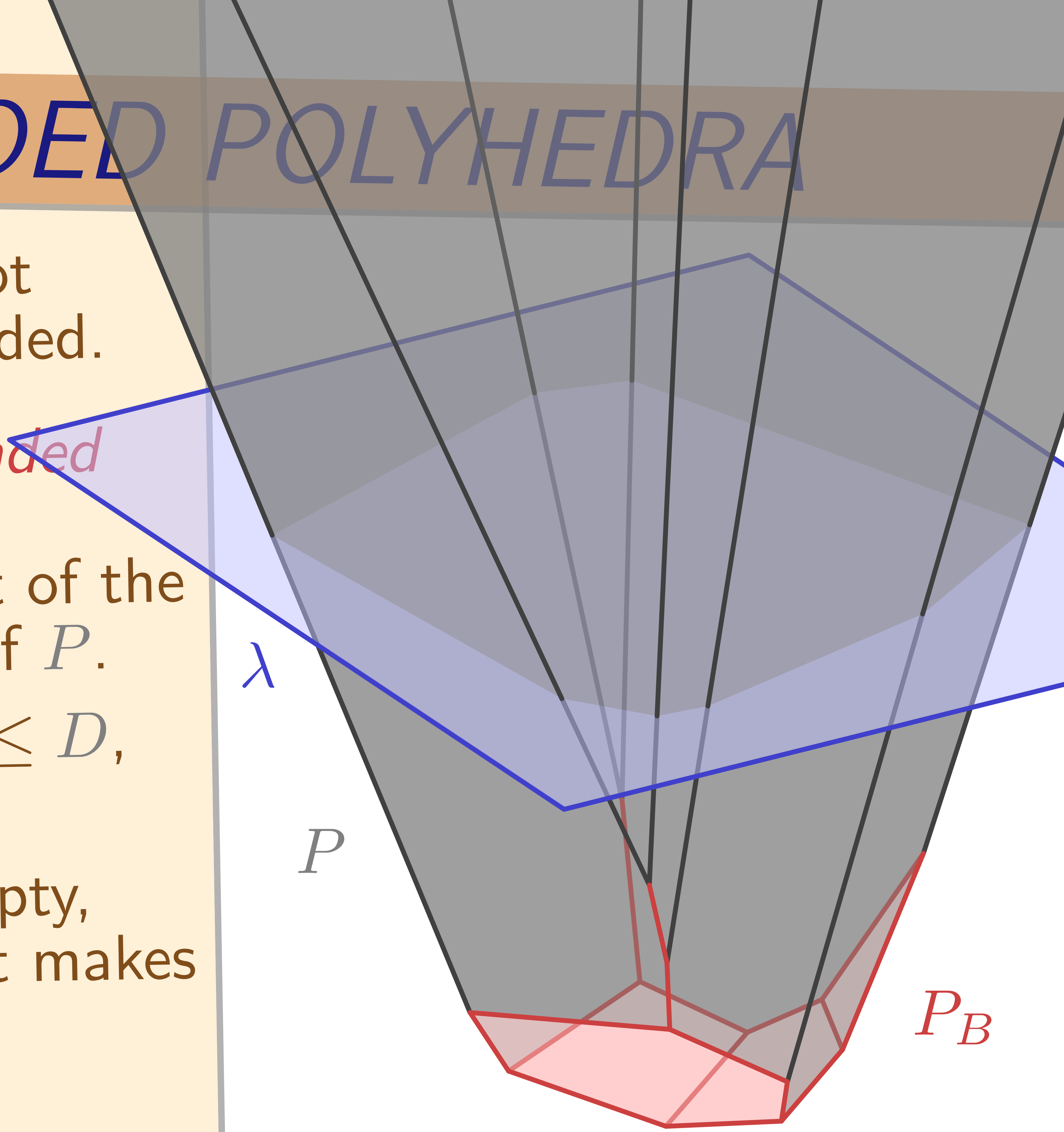

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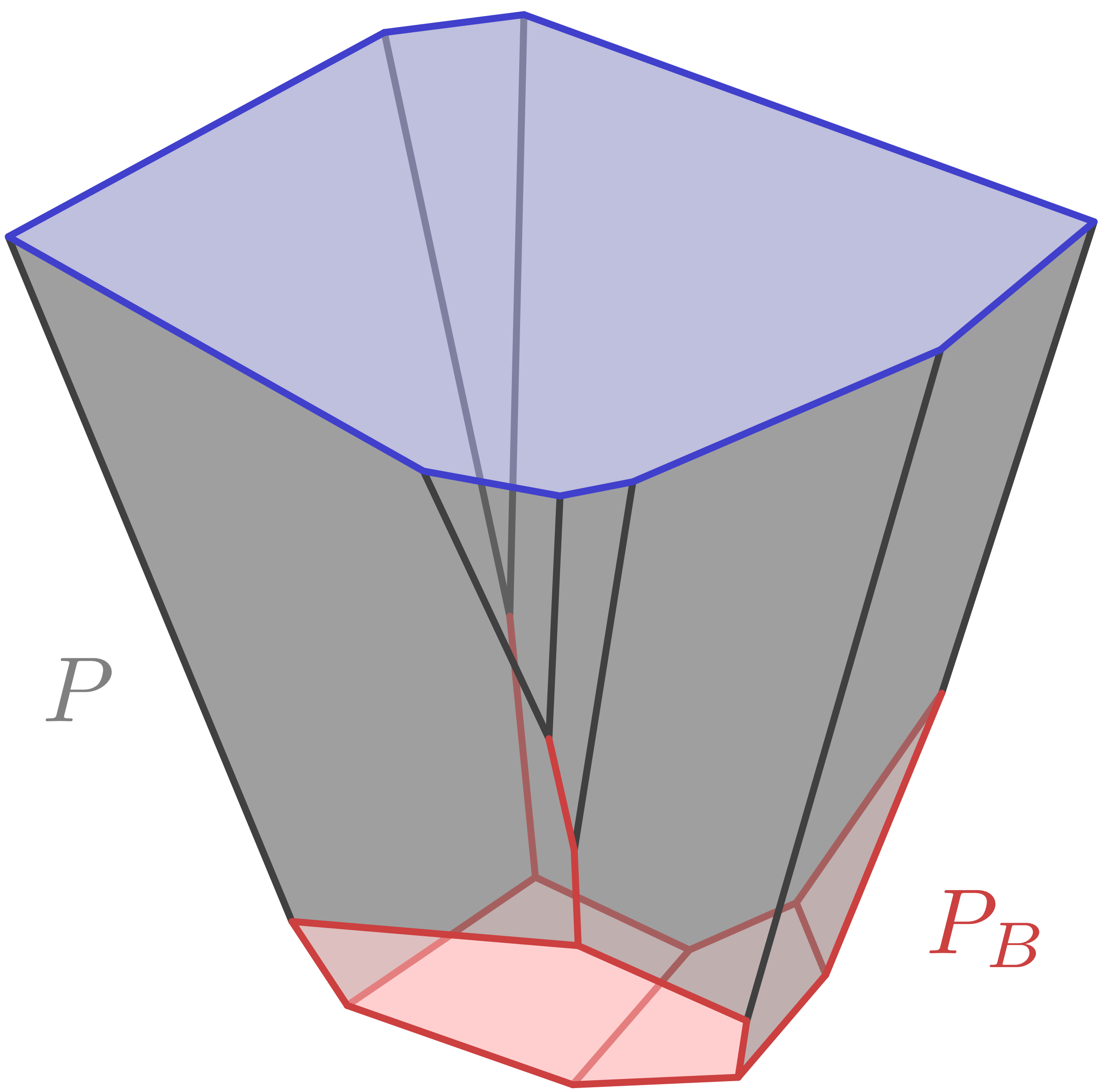

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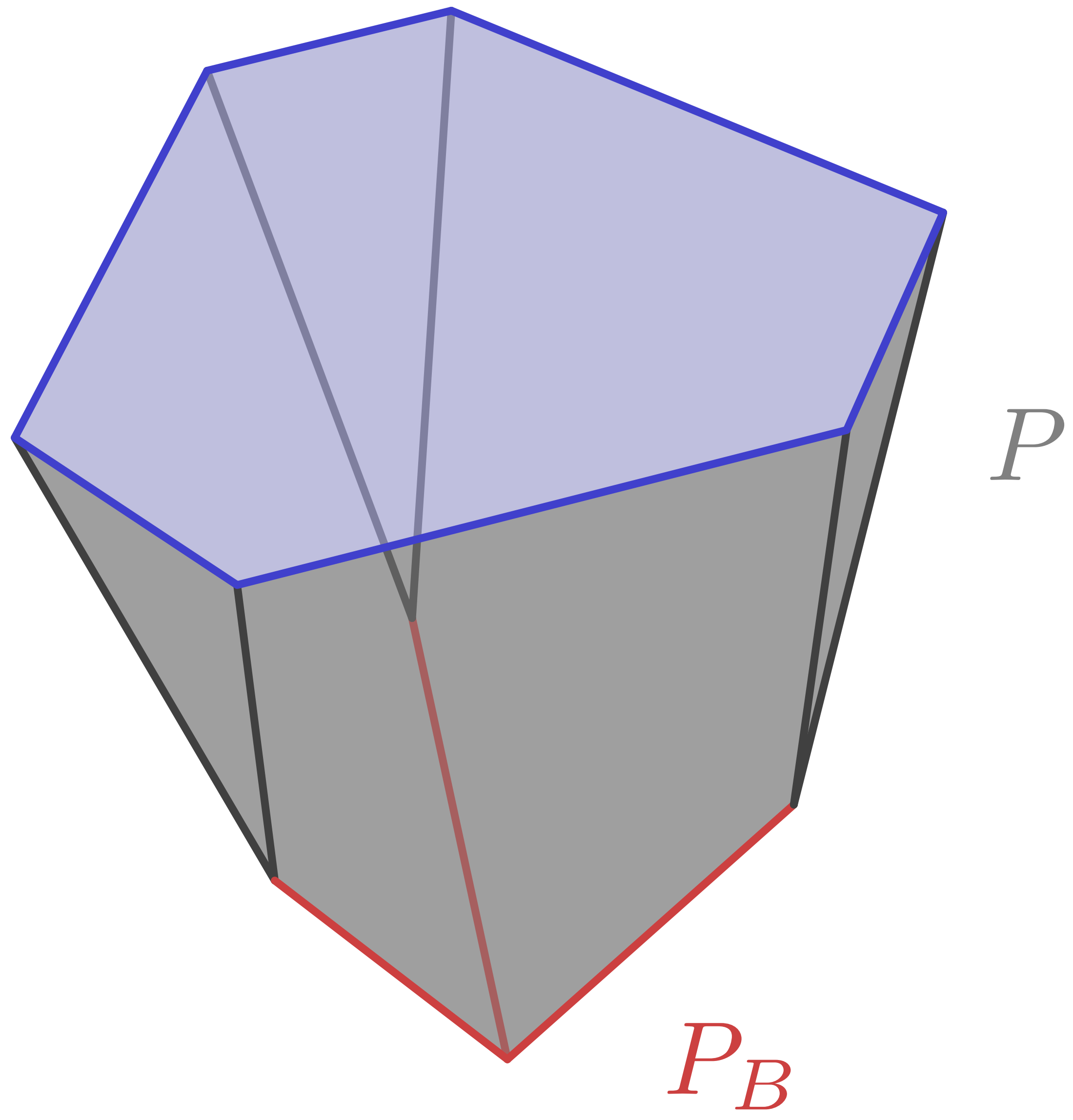

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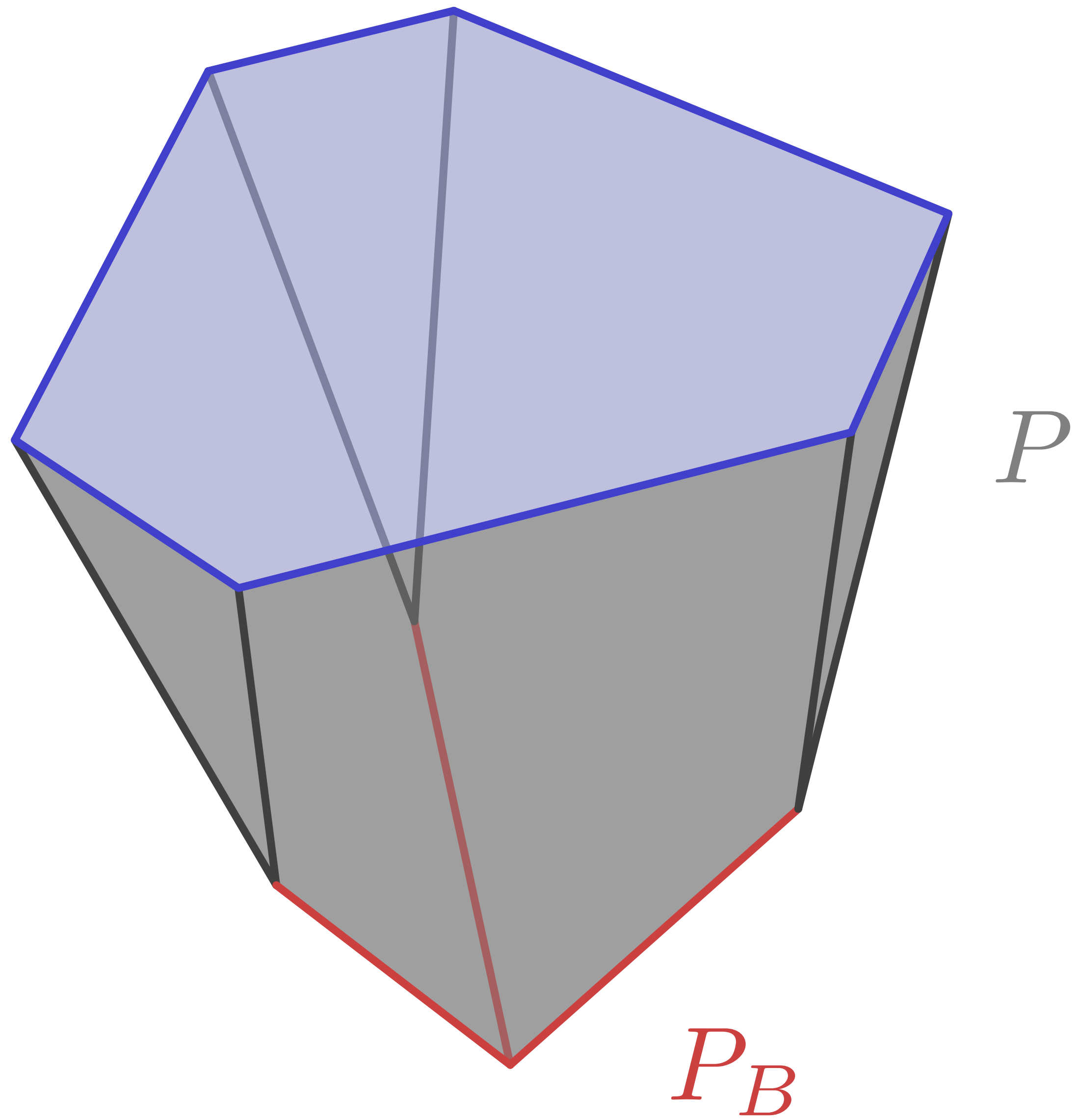

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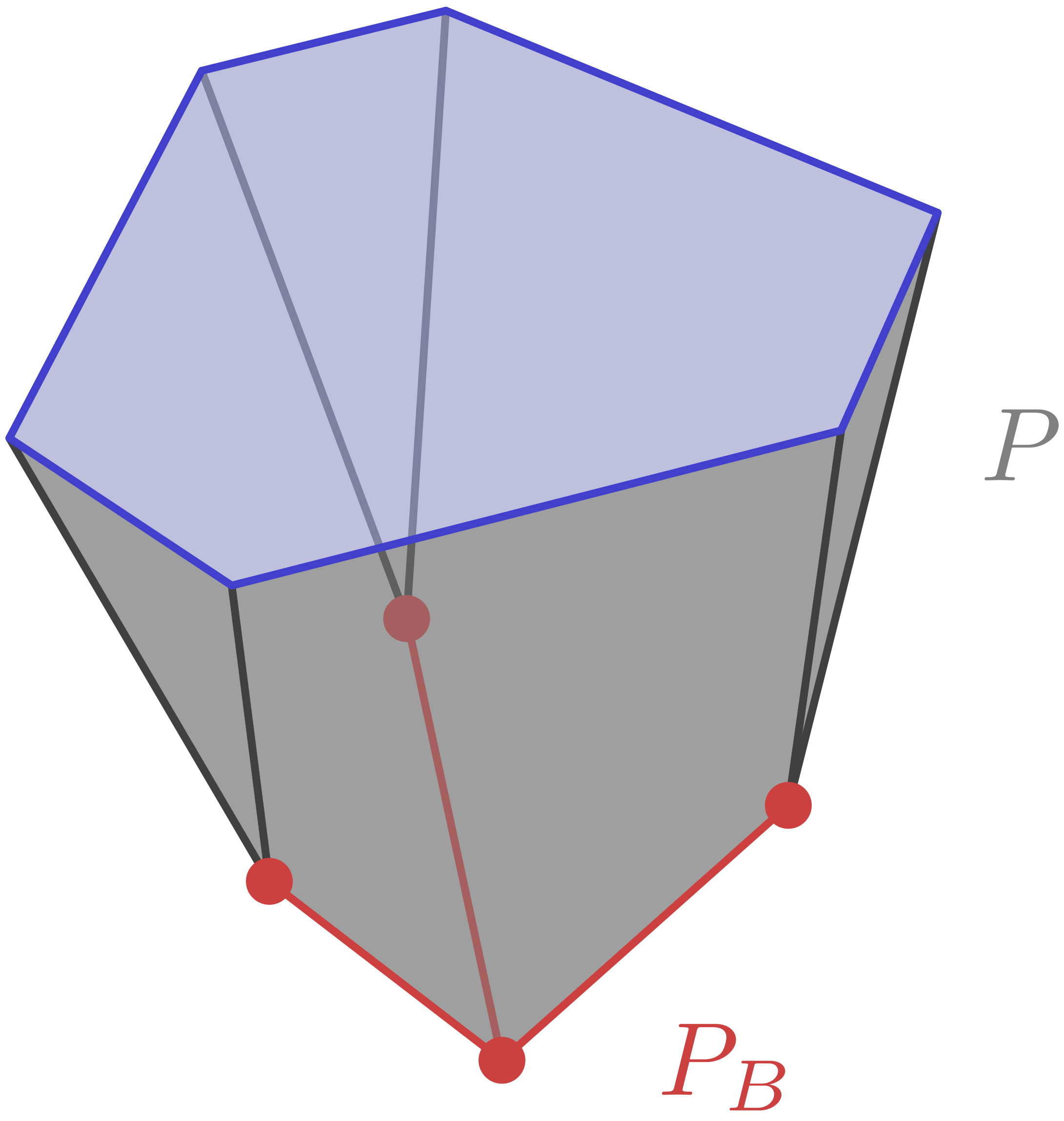

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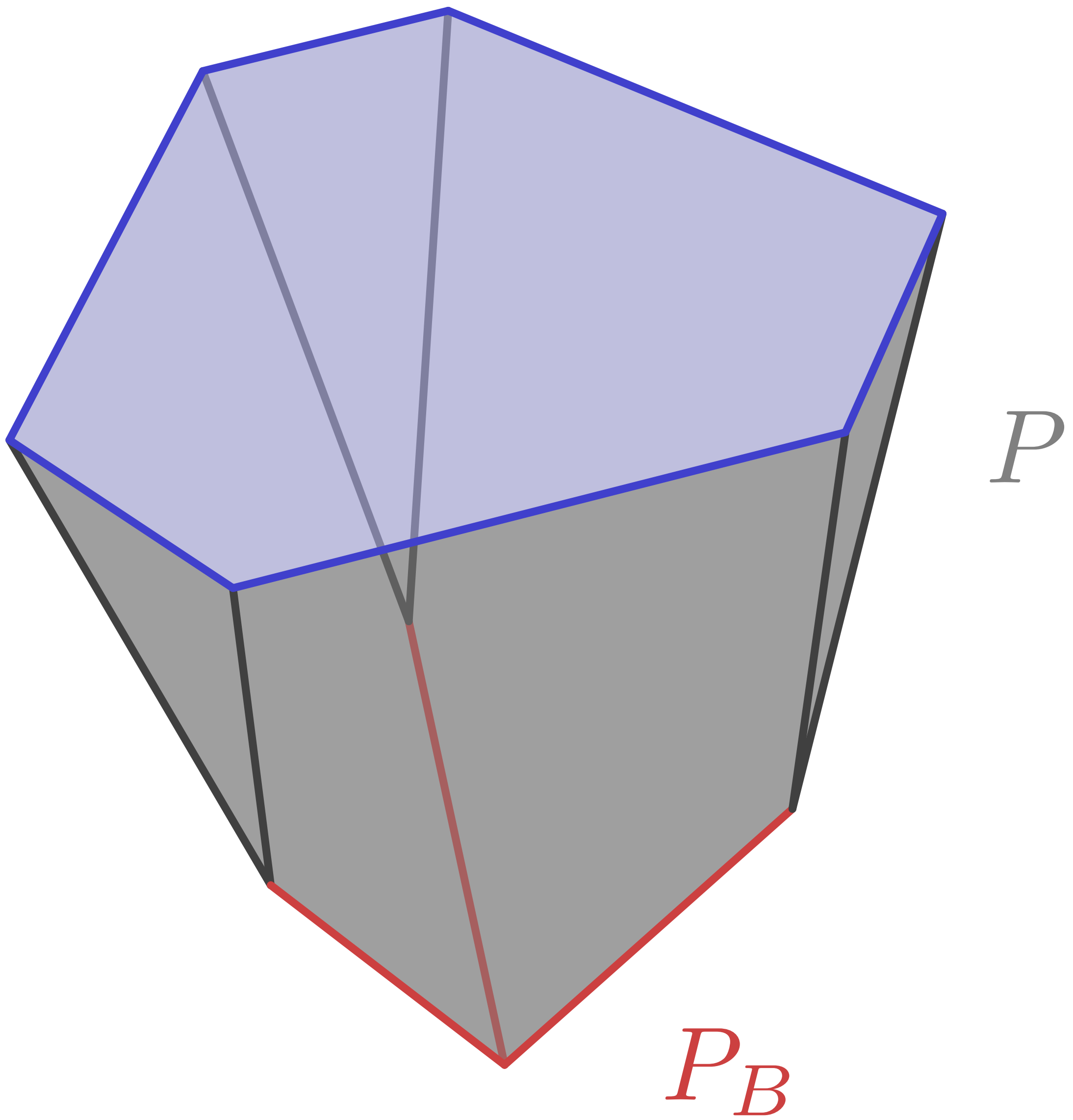

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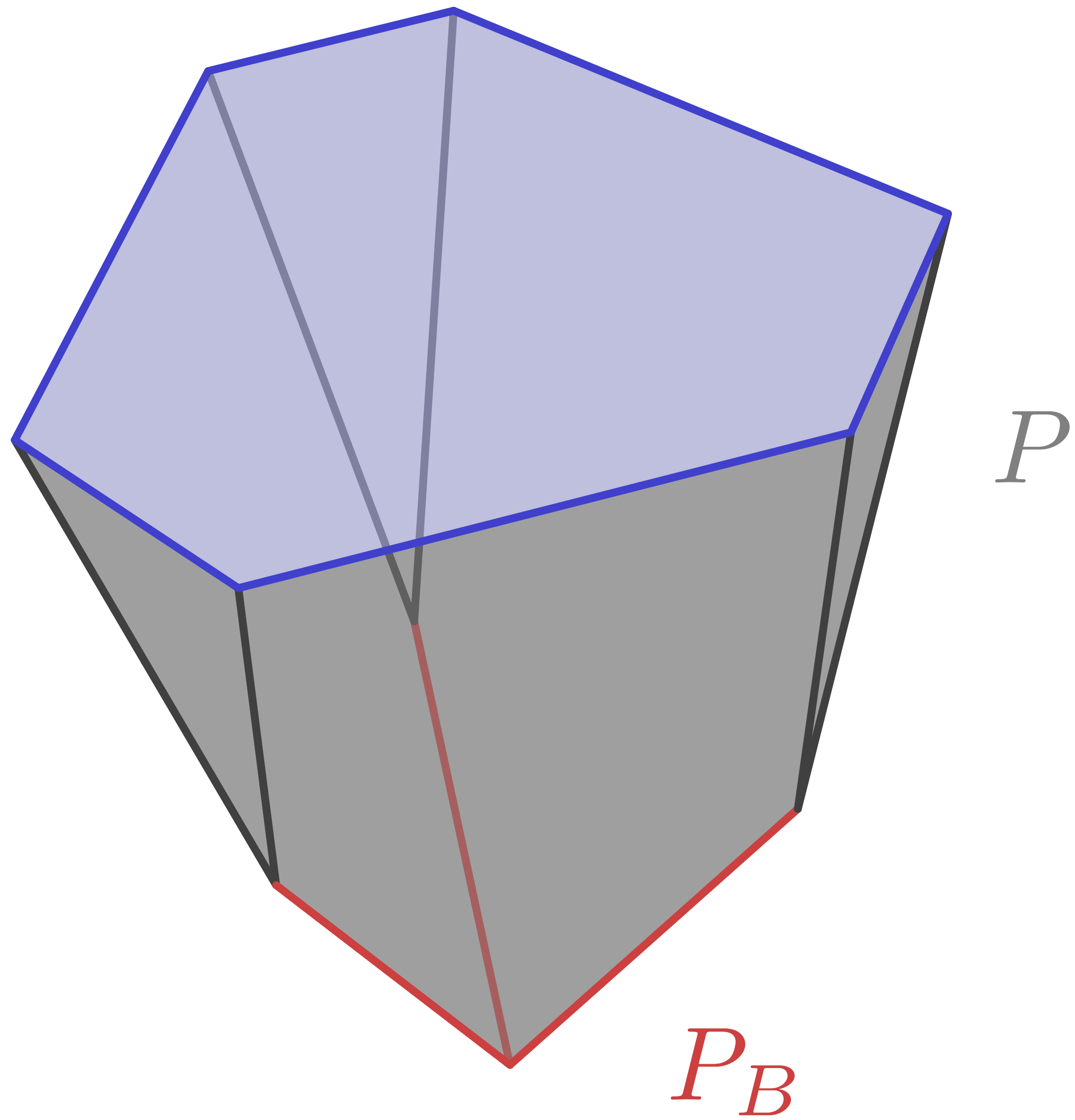

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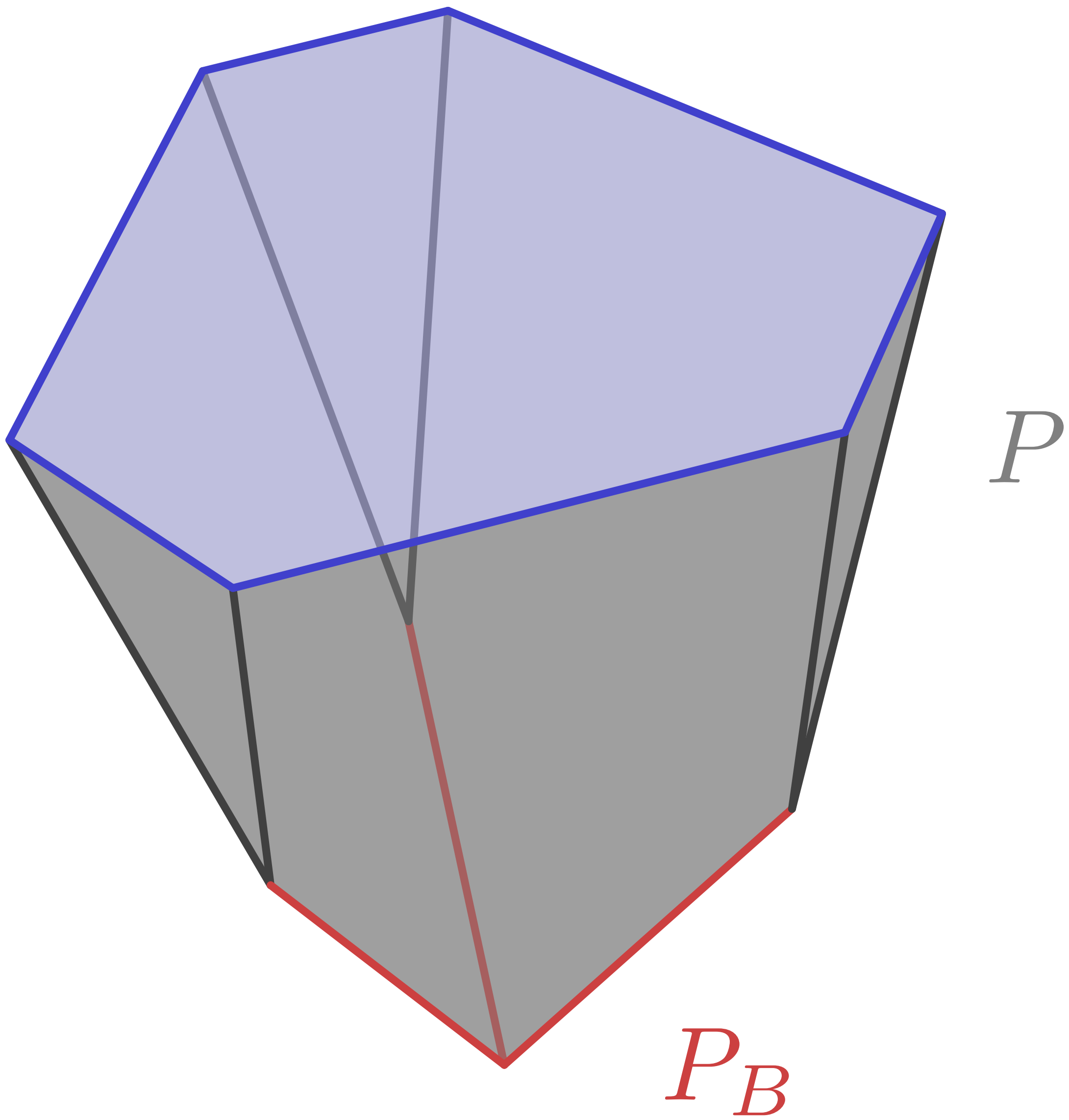
P_B can be computed in $O(n^d L + n^{2d^2+3d})$ or $O(n^{d^2+d} L)$ time.



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Given P and λ,
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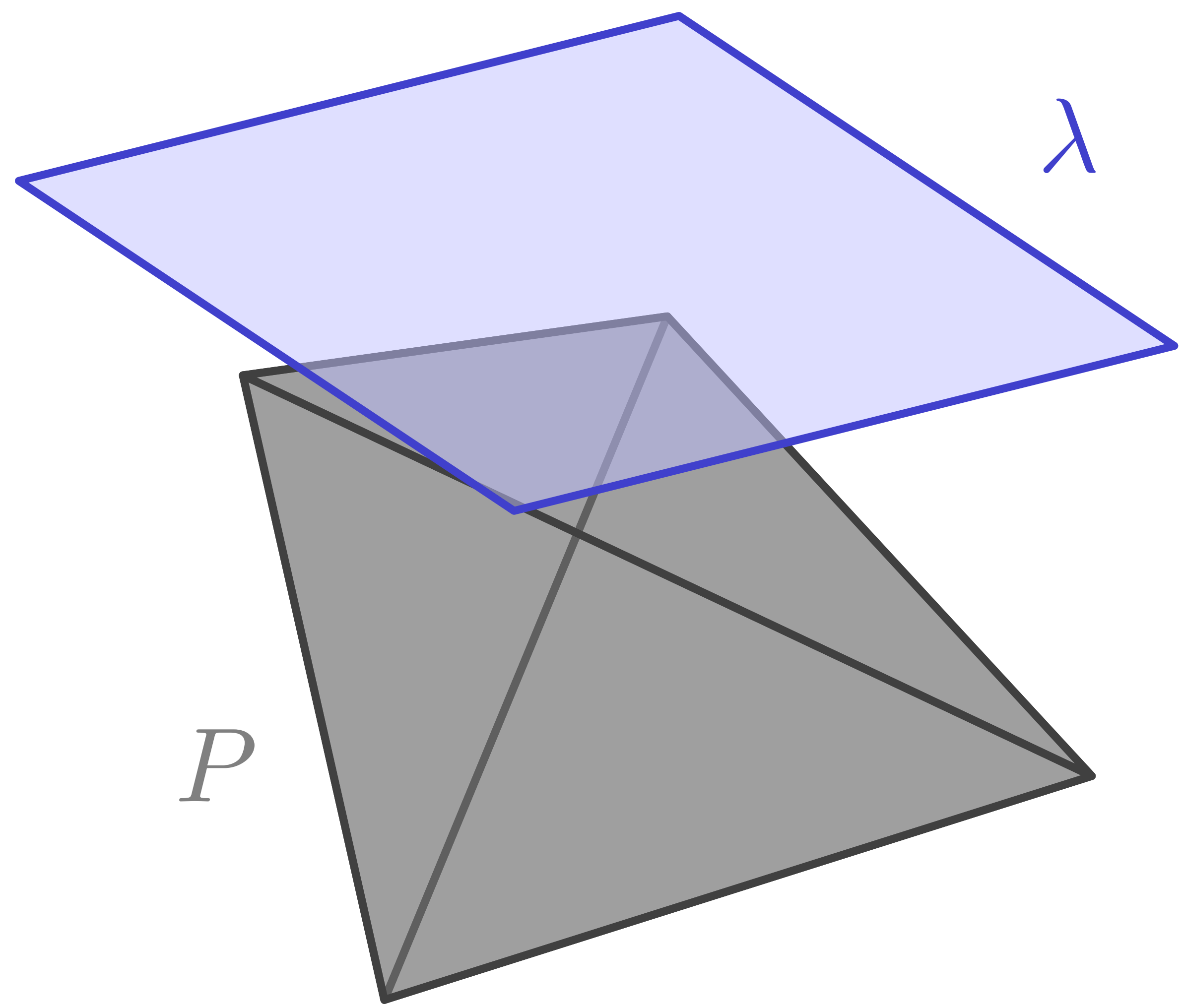

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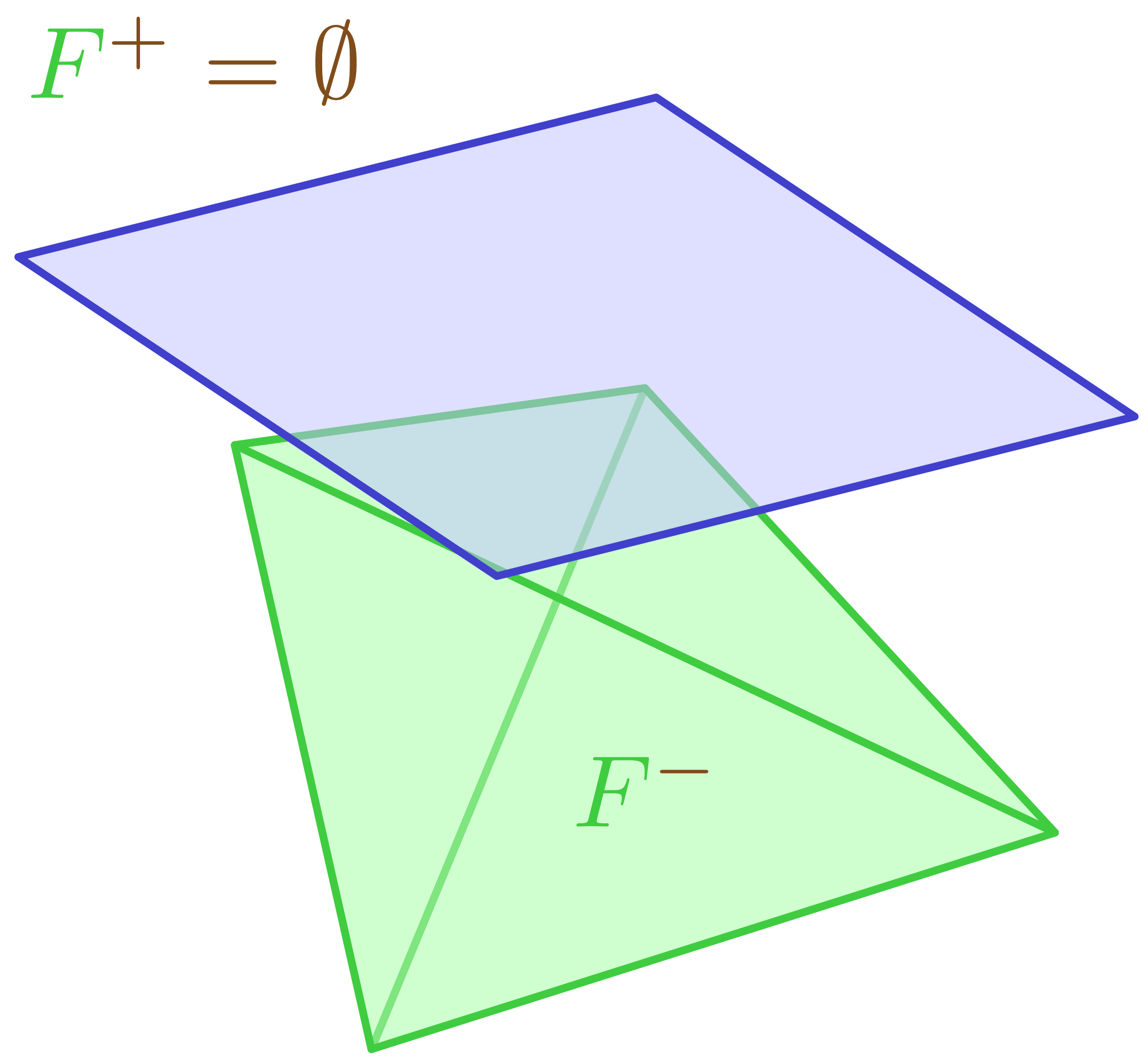

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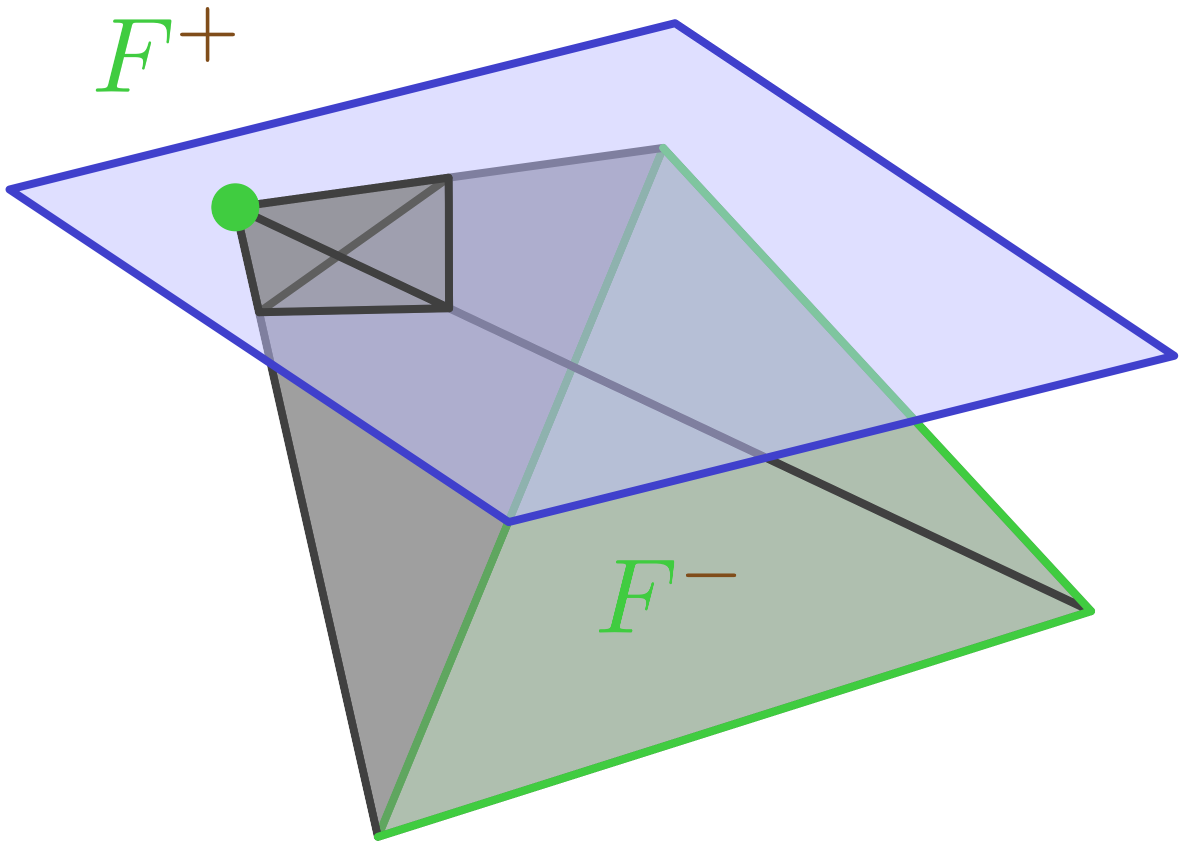

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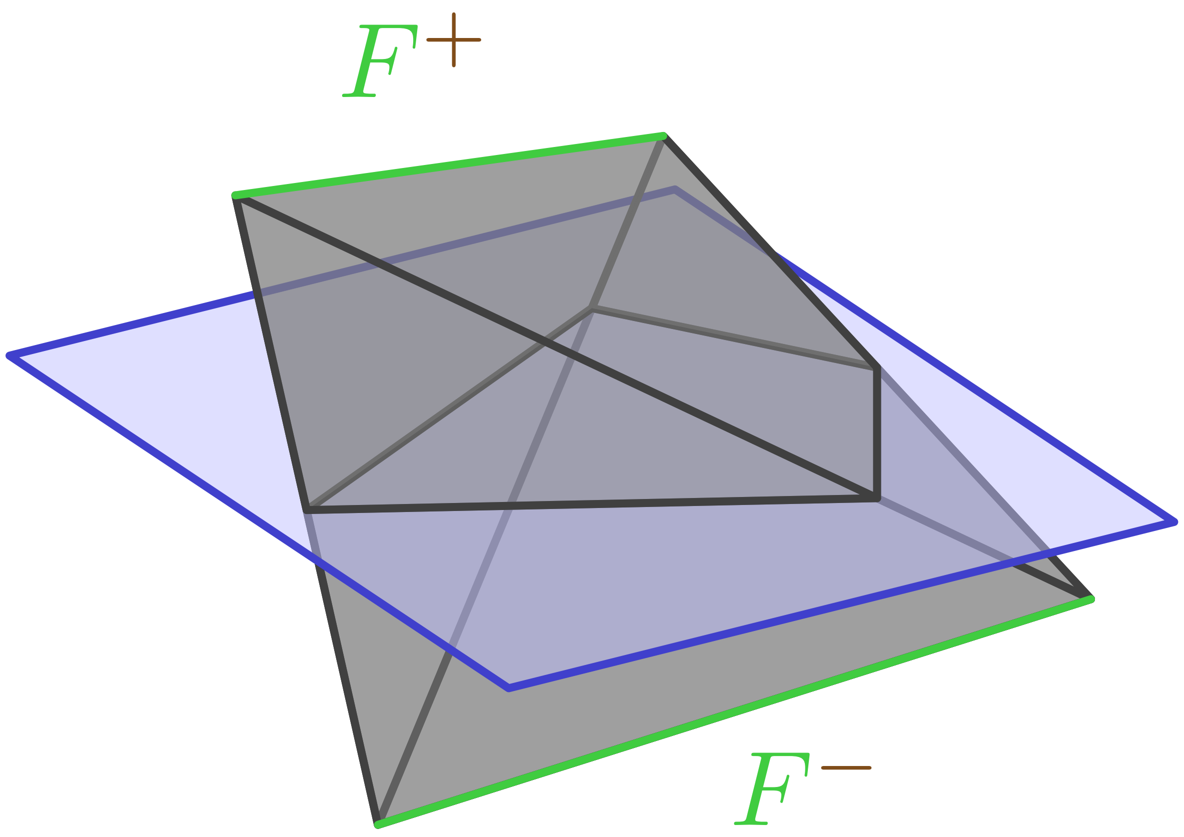

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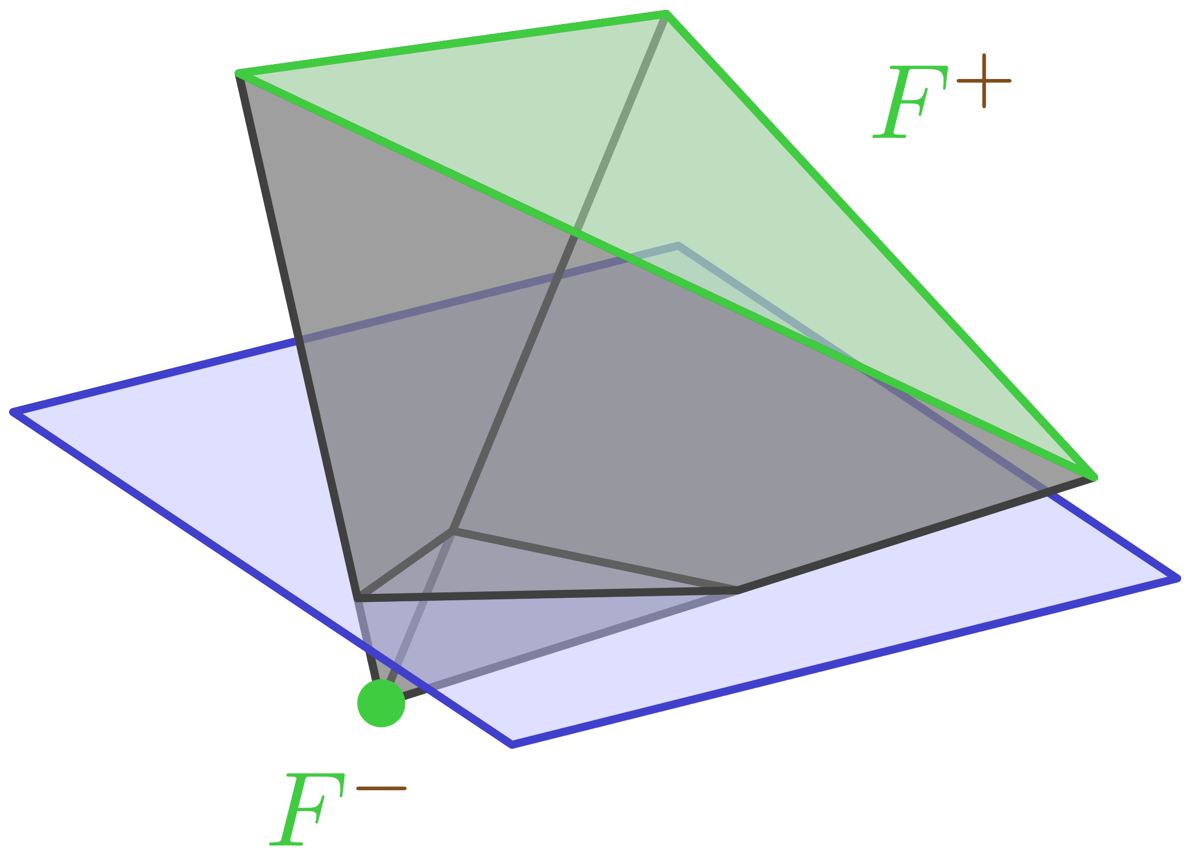

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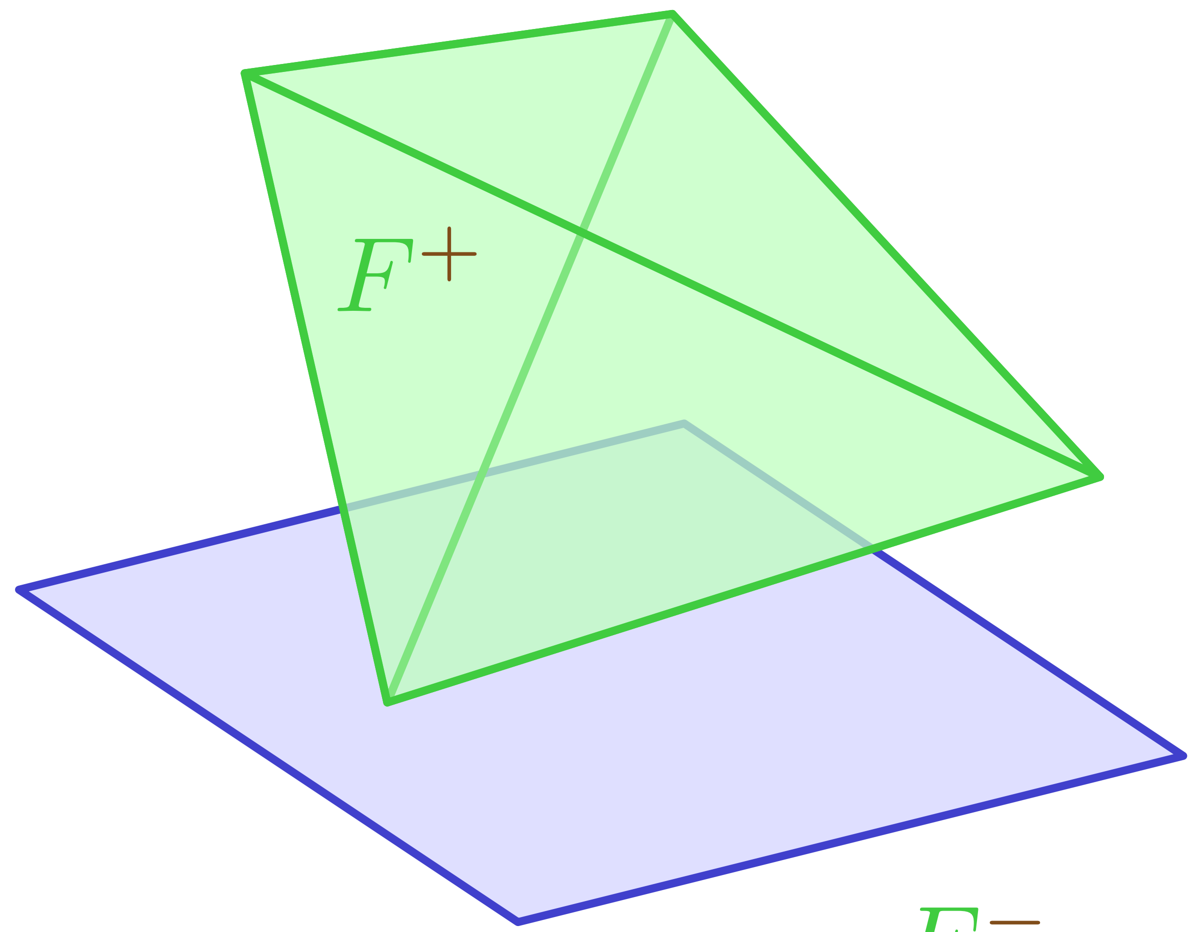

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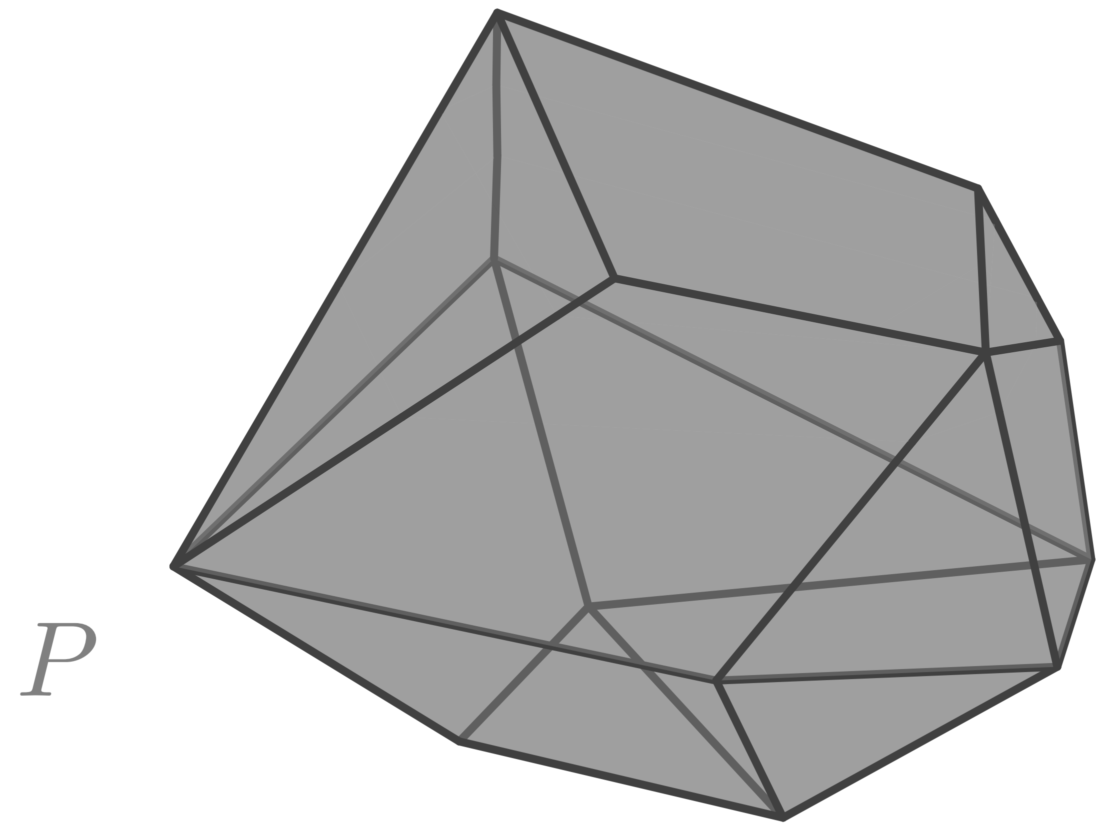

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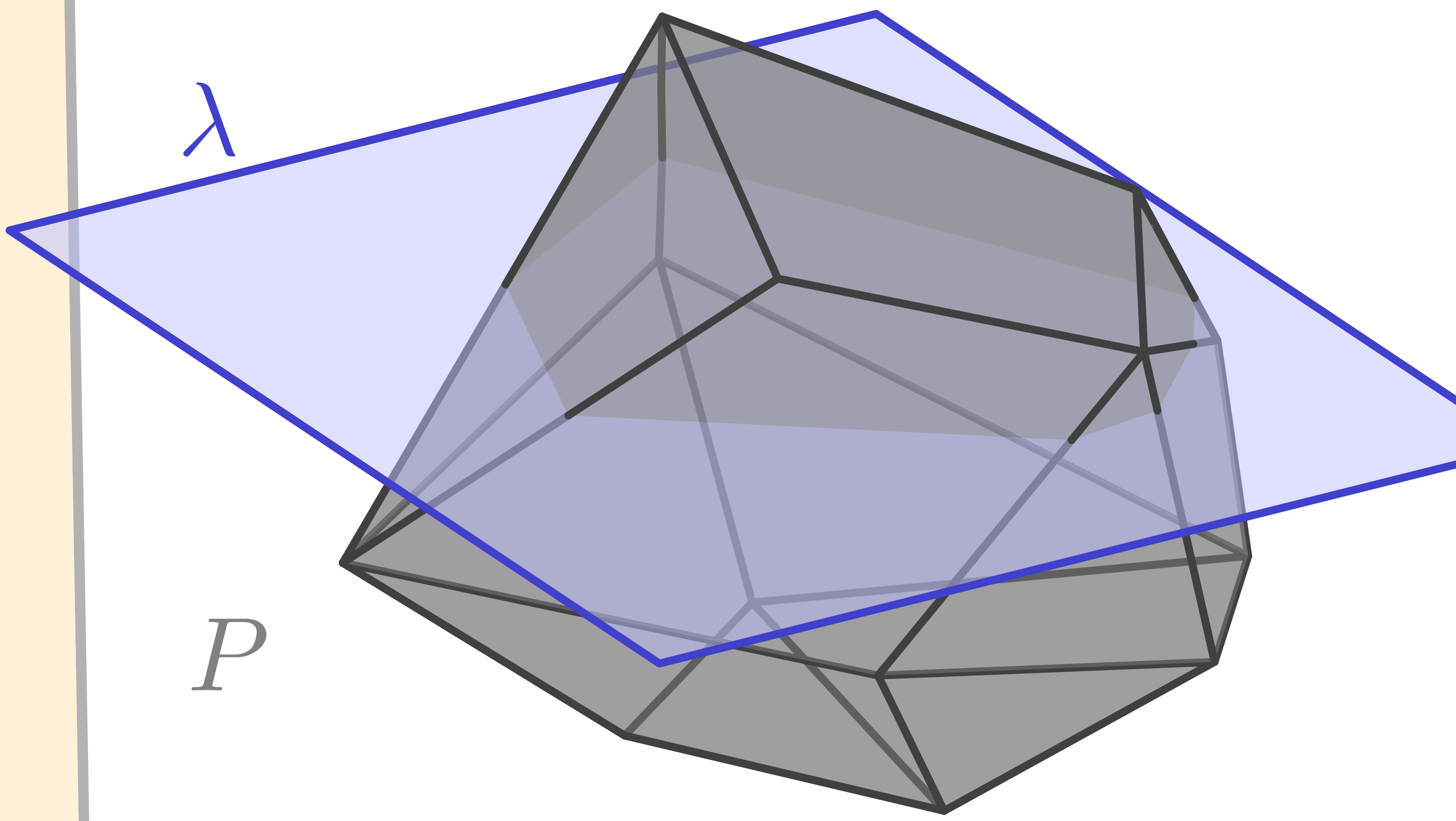

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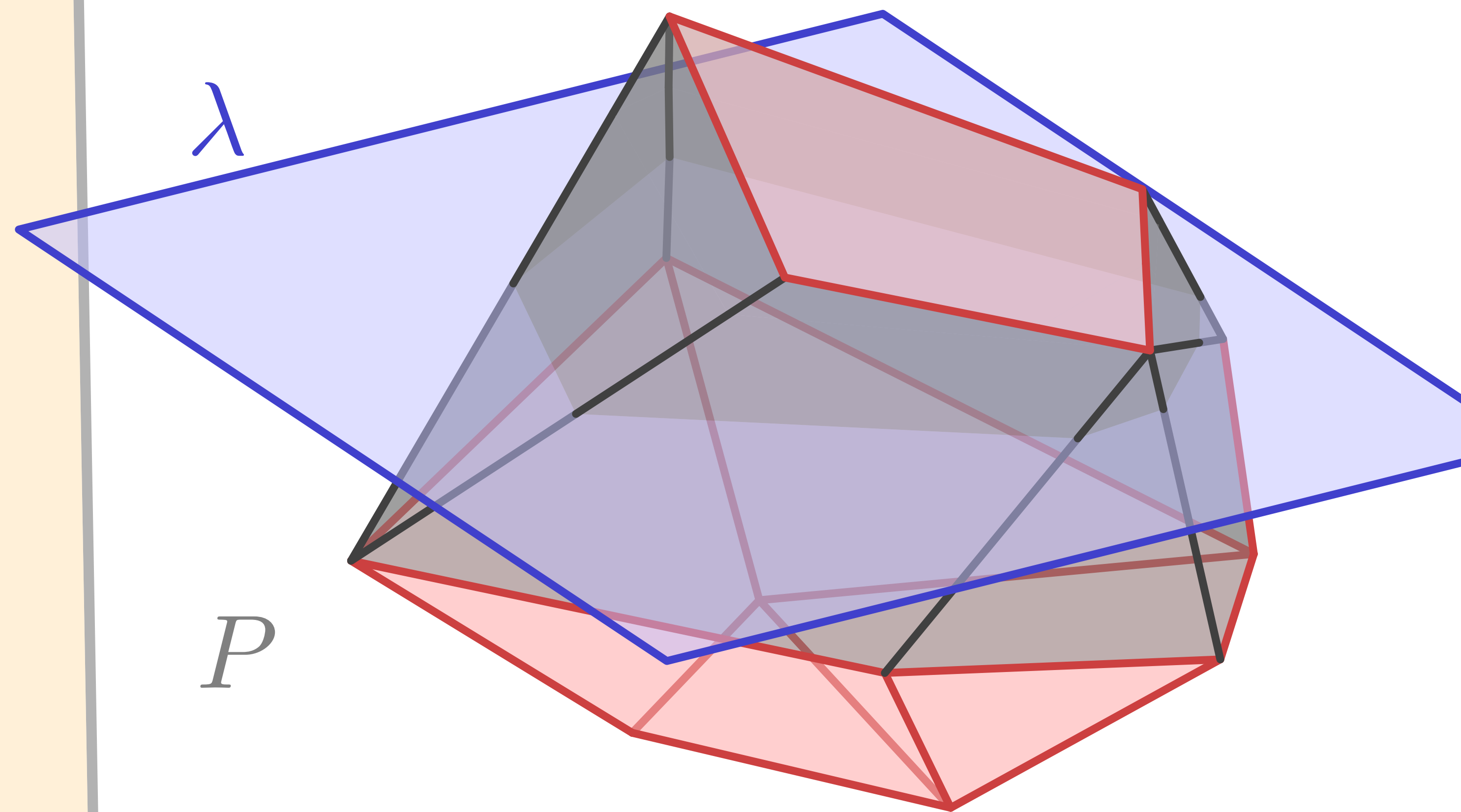

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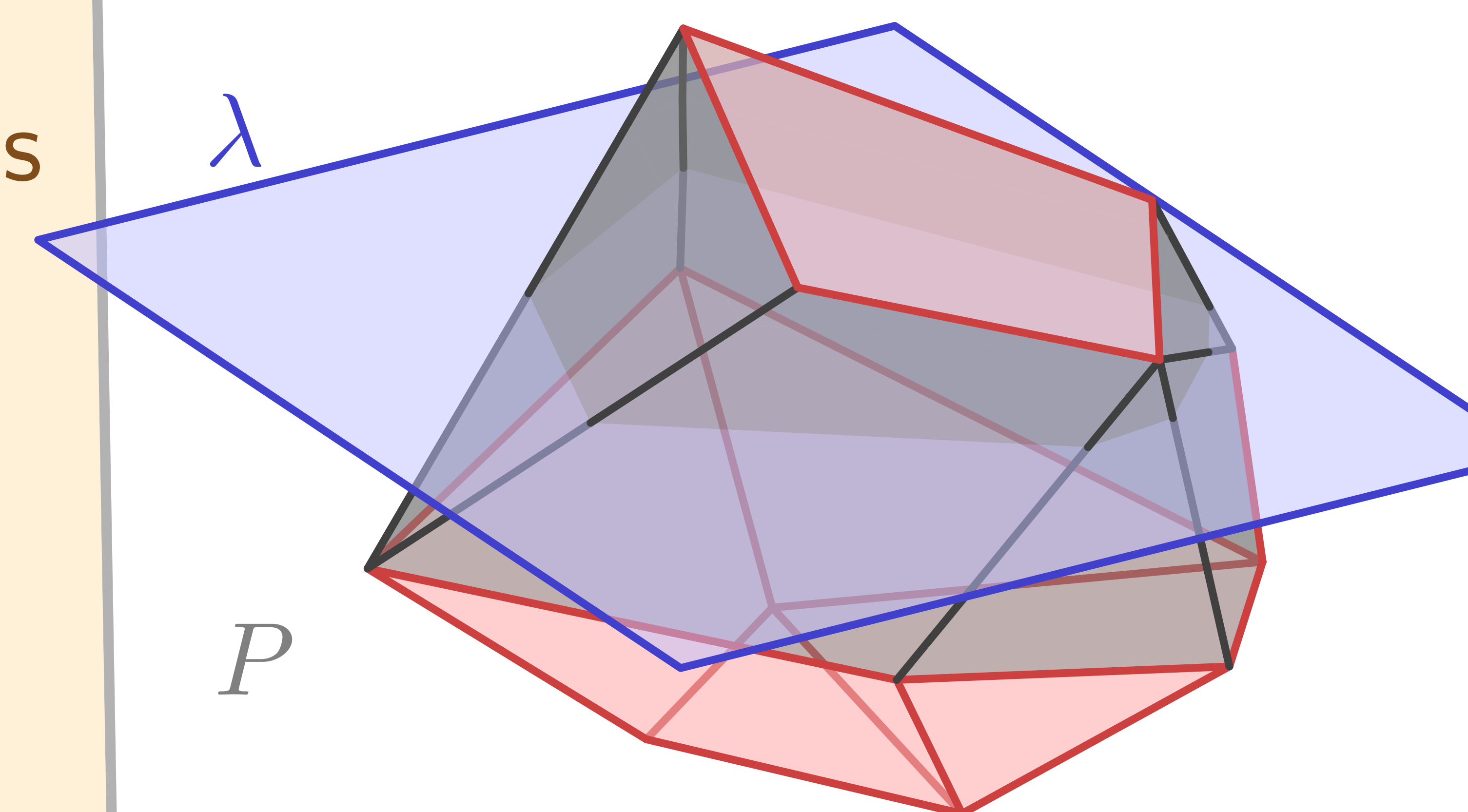

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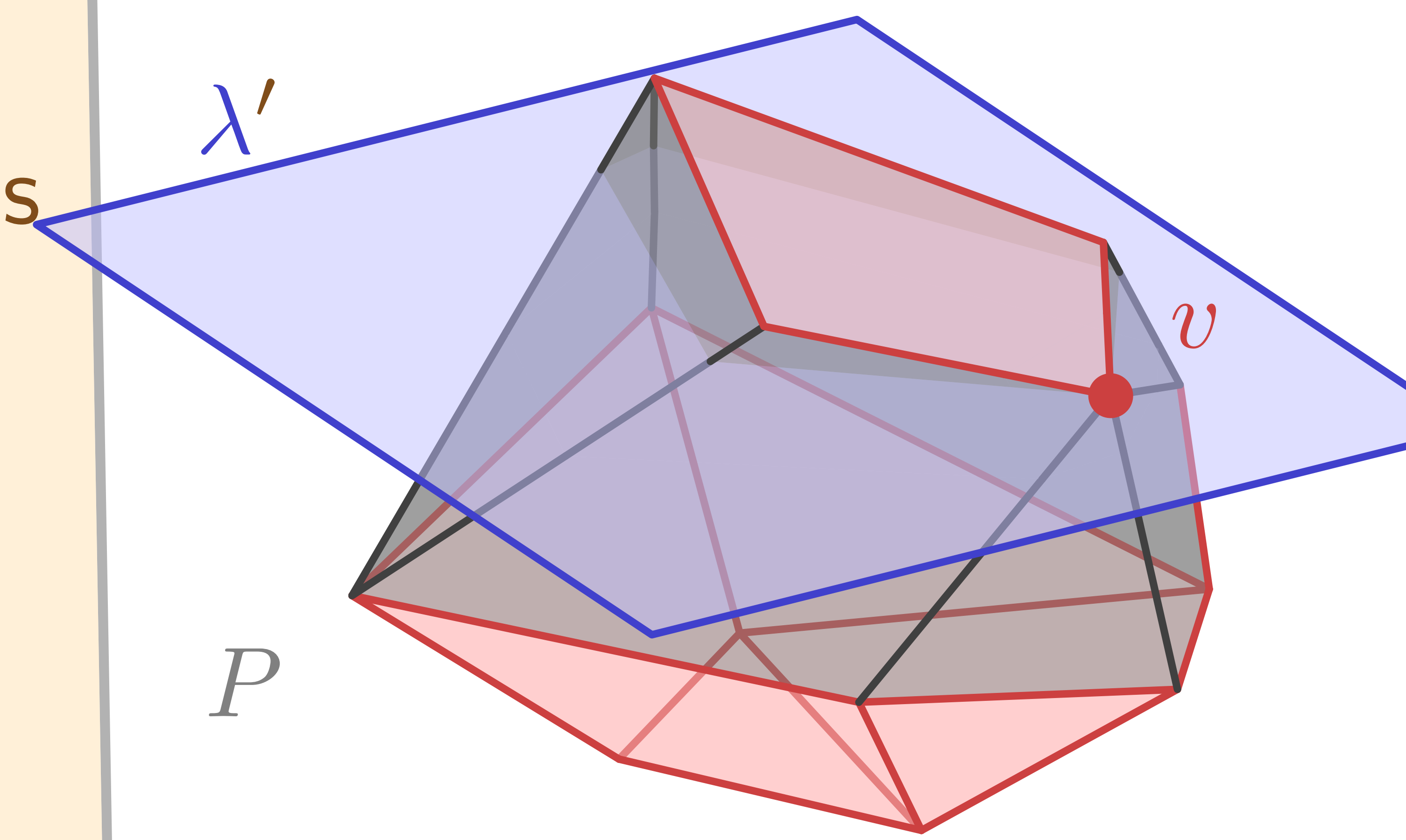

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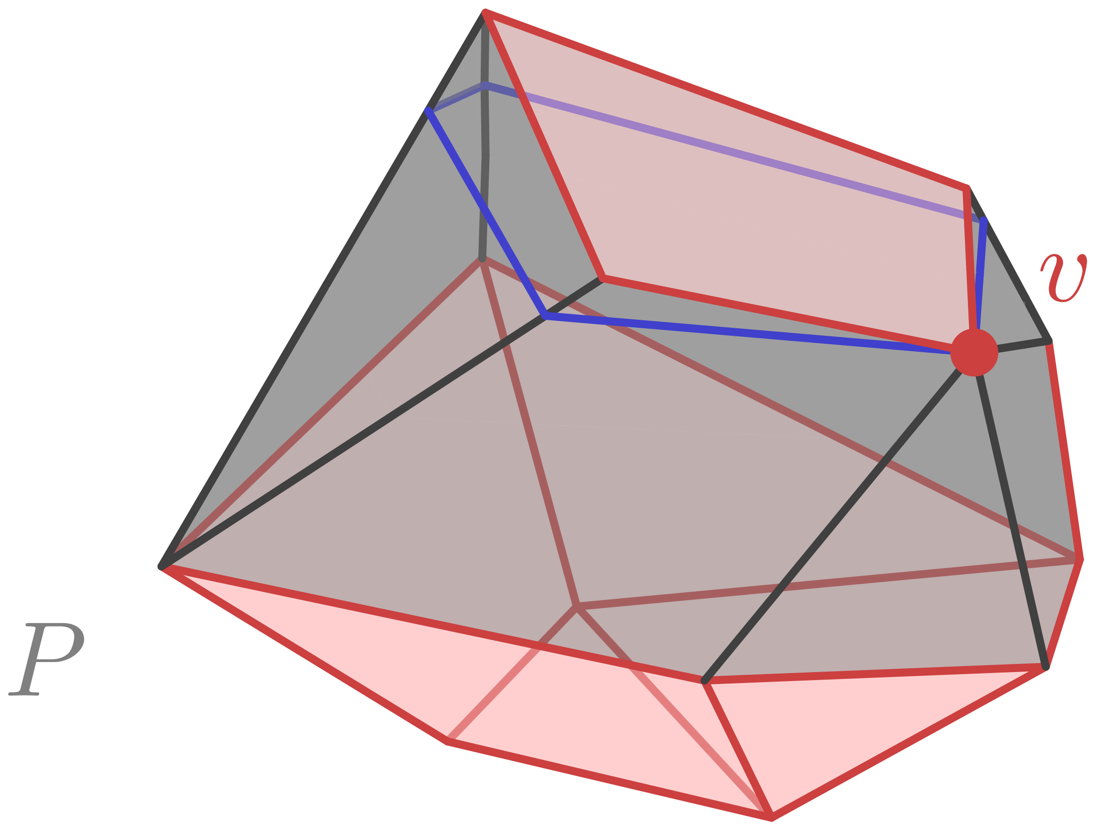

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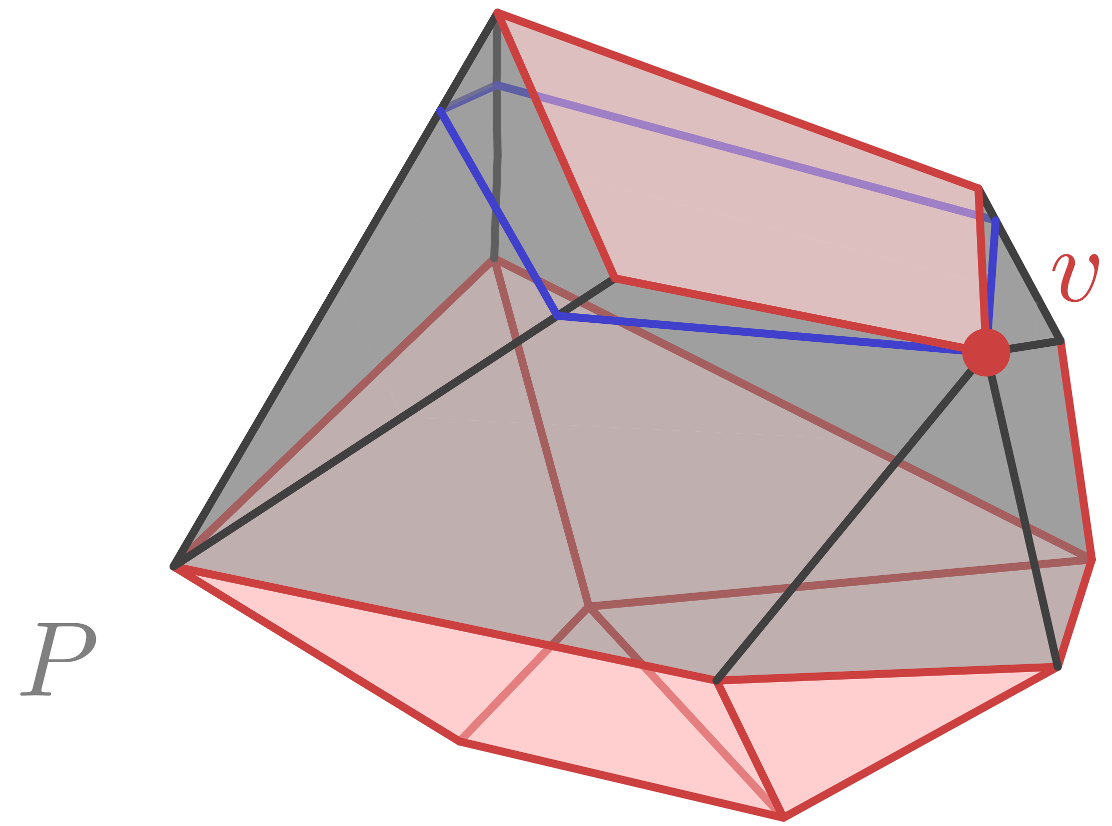

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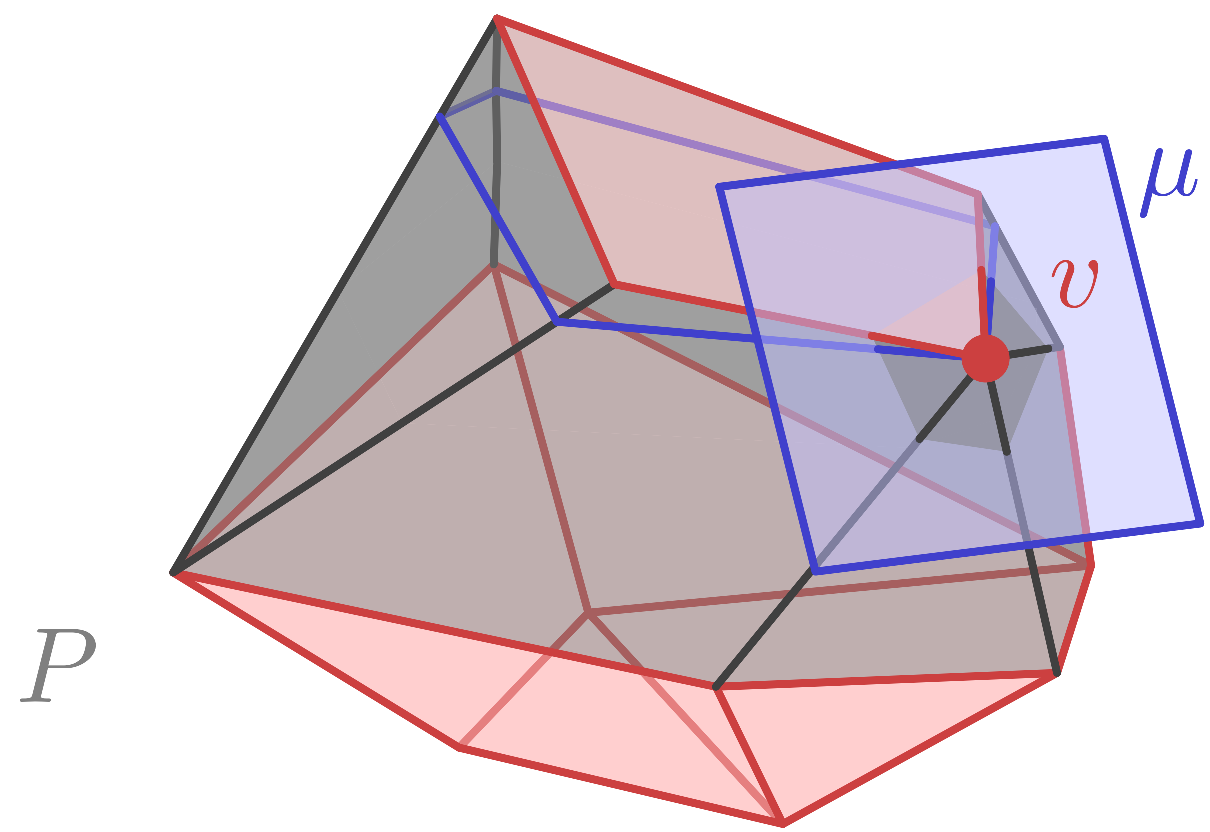

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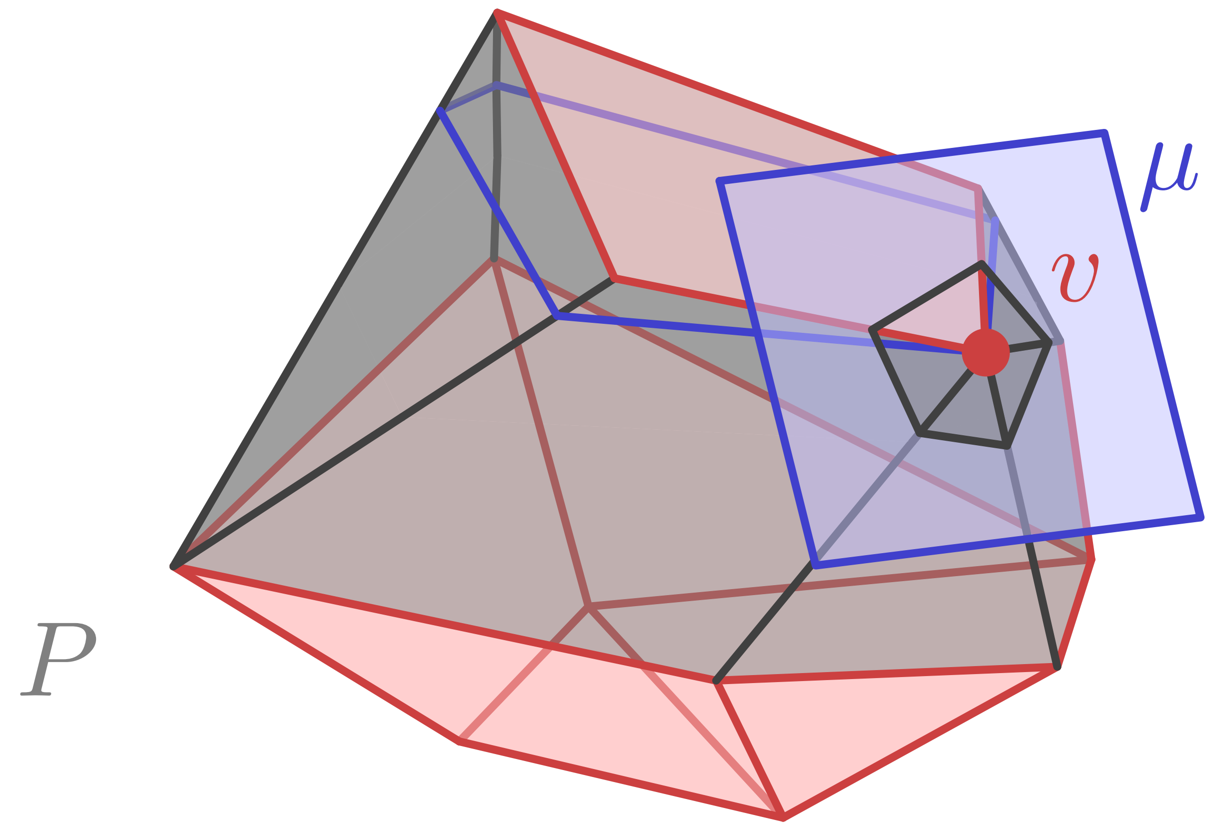

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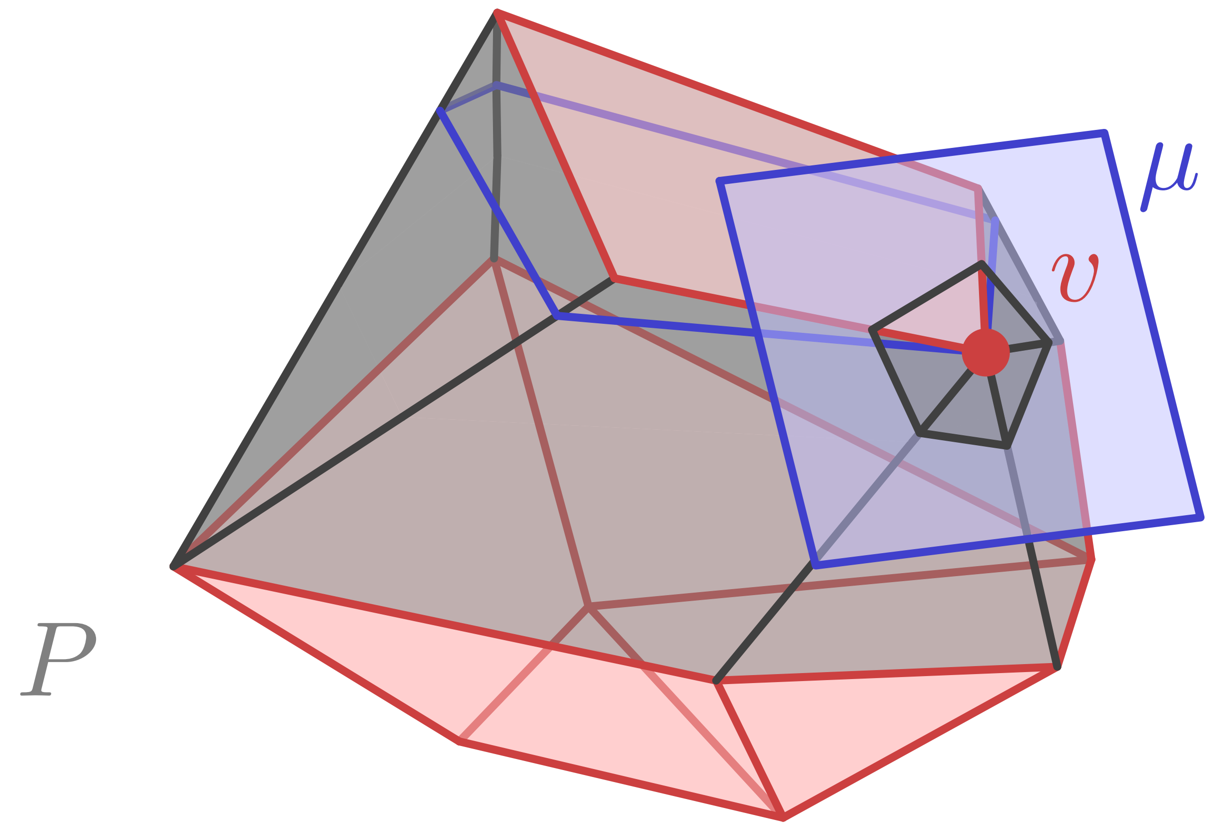

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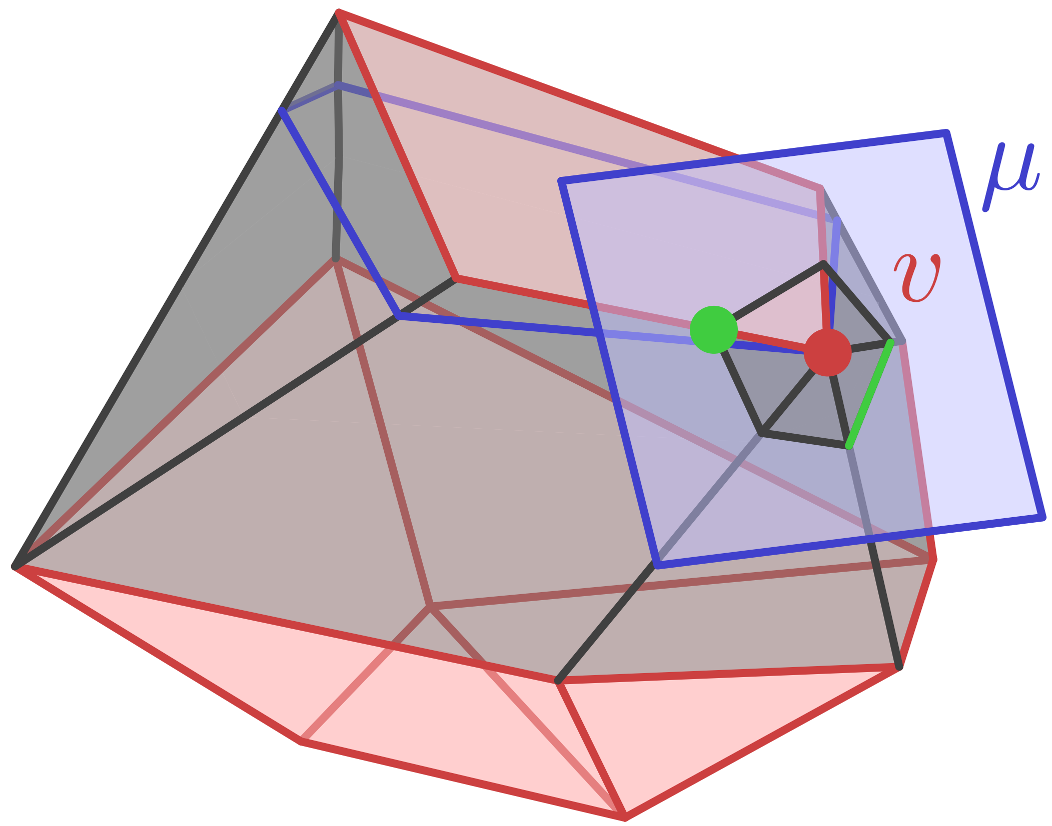

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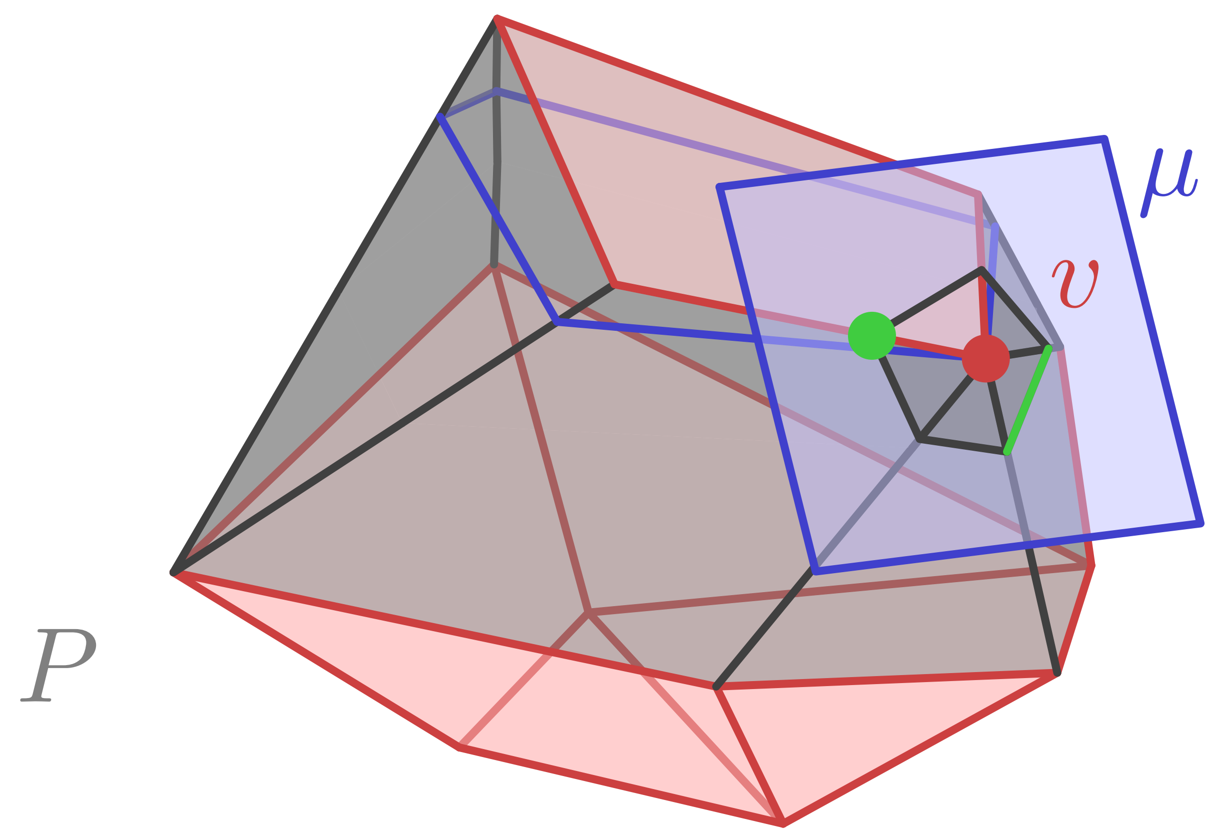

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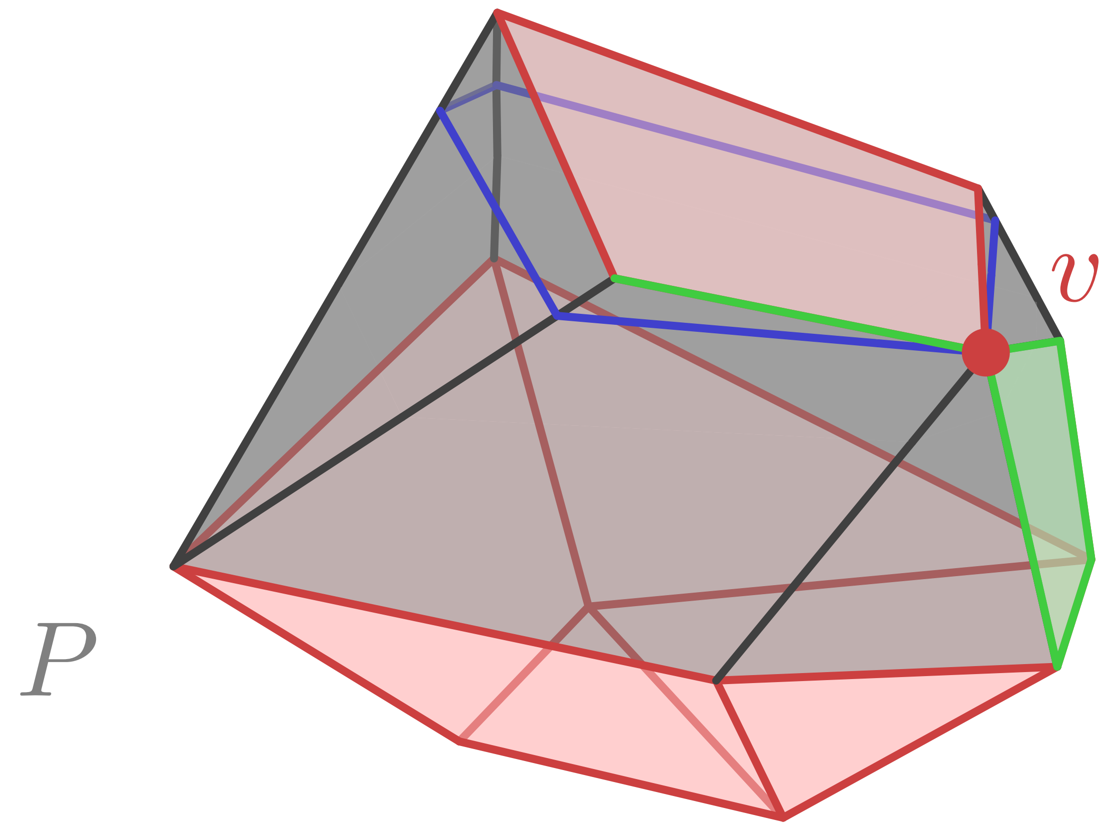

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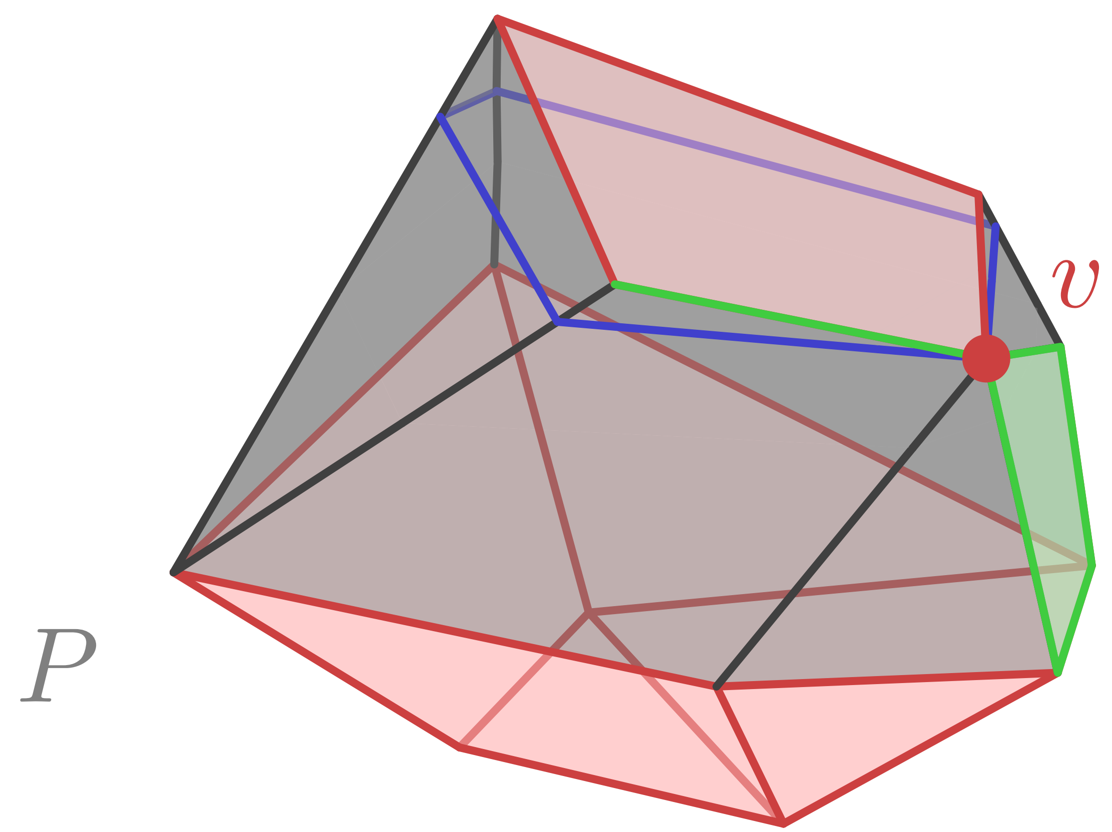

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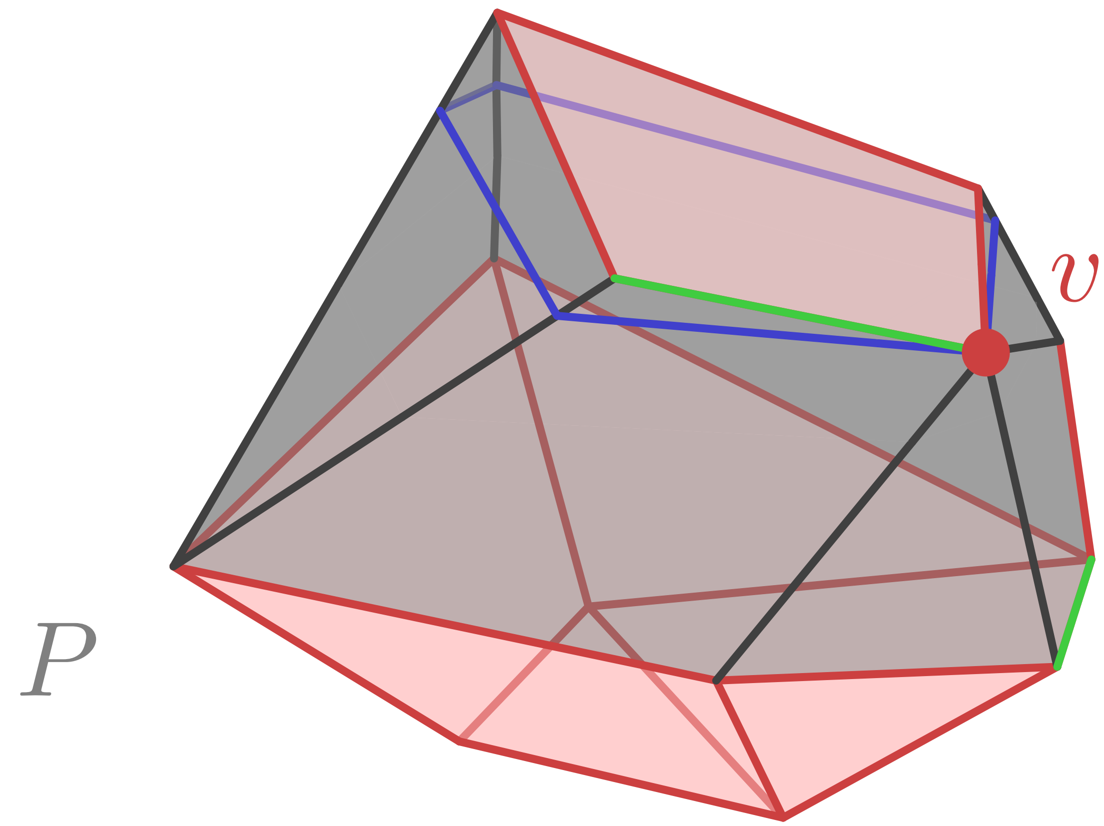

SPLITTING LEMMA

there are two faces $F^+, F^- \subset P$, one of each side of λ, such that $\mathrm{aff}(F^+ \cup F^-) = \mathrm{aff}(P)$.

Proof
Use induction on $\dim(P)$.

Move λ up until it hits the first vertex v.

Let μ slice v from P.

Find F^+_* and F^-_*.

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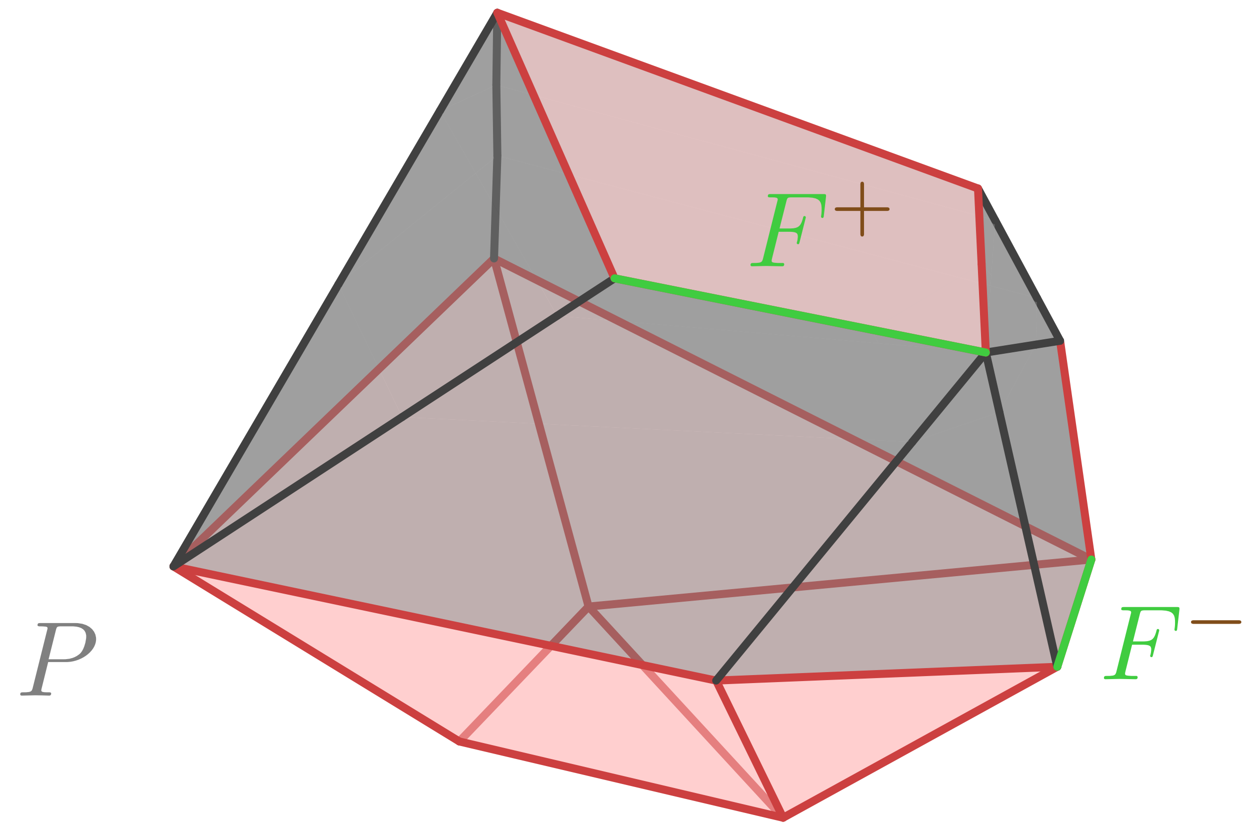

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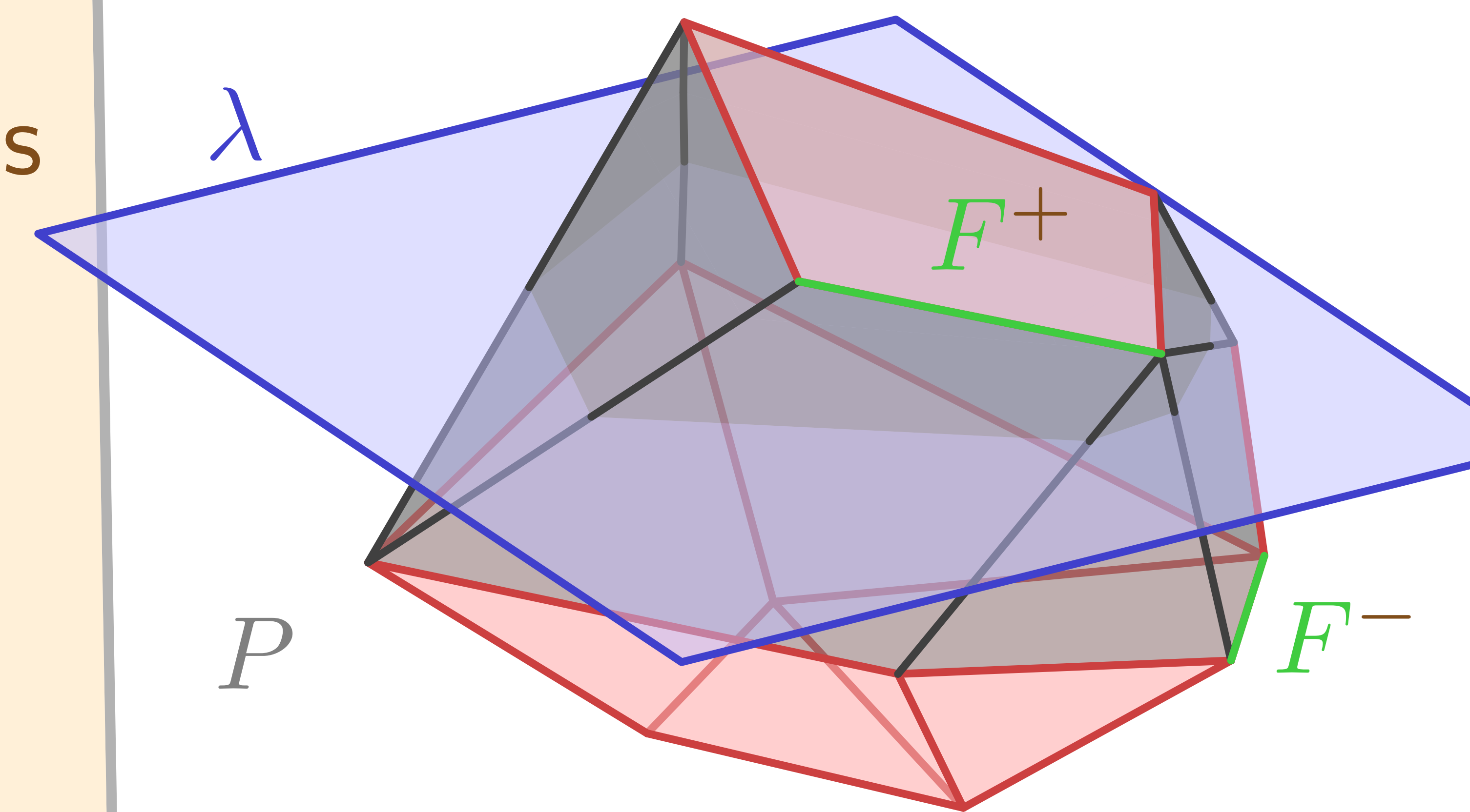

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Given P and v,
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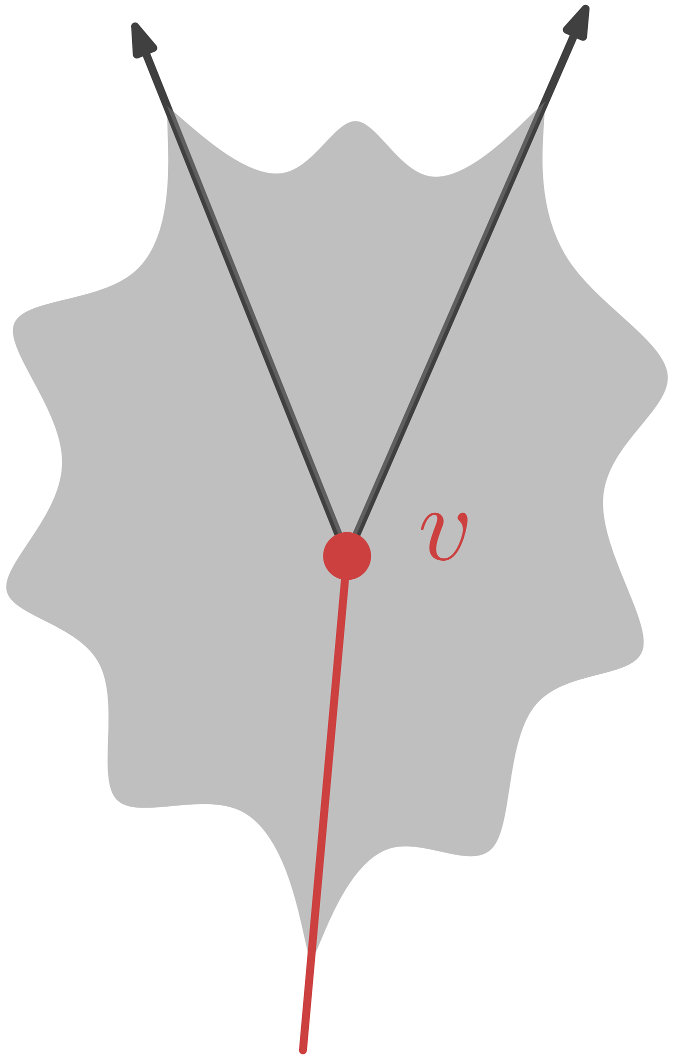

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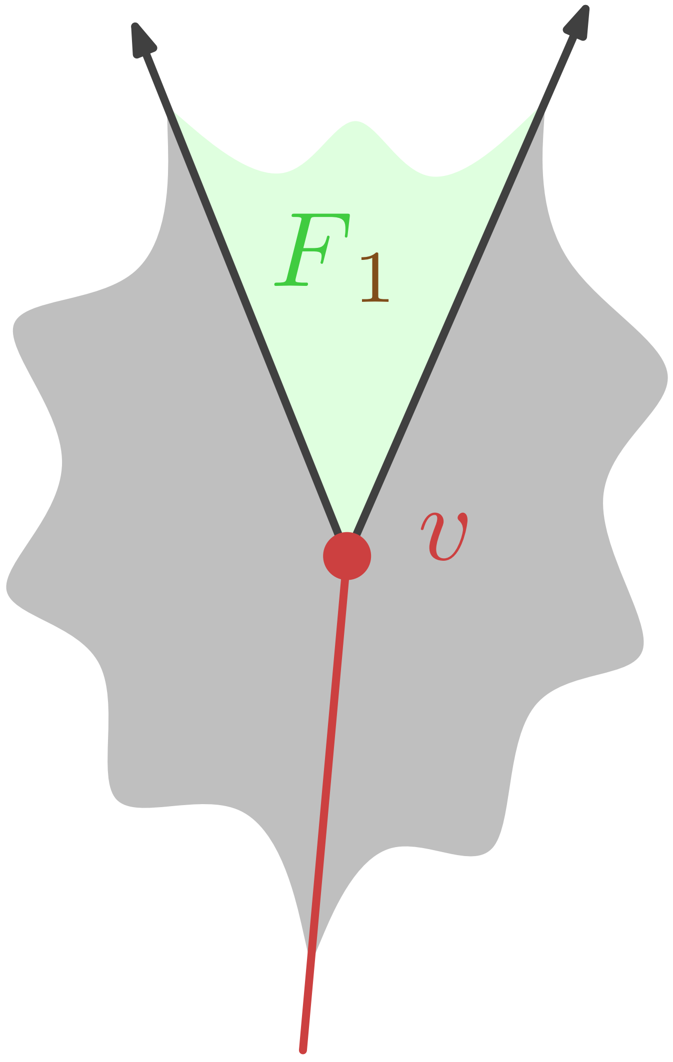

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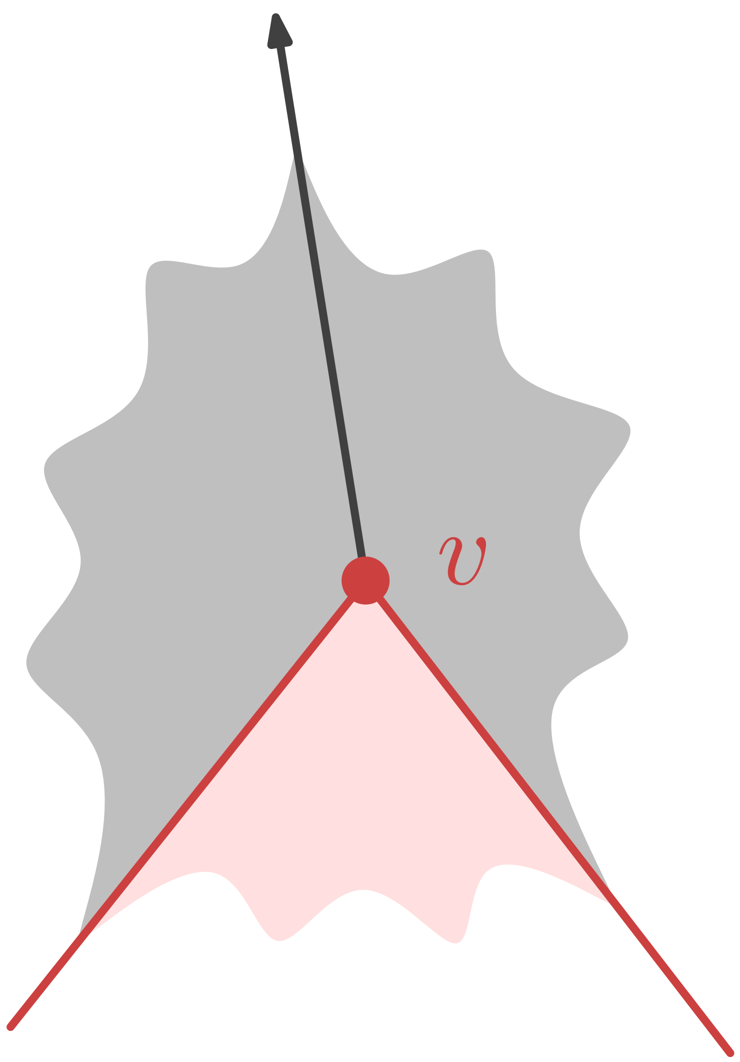

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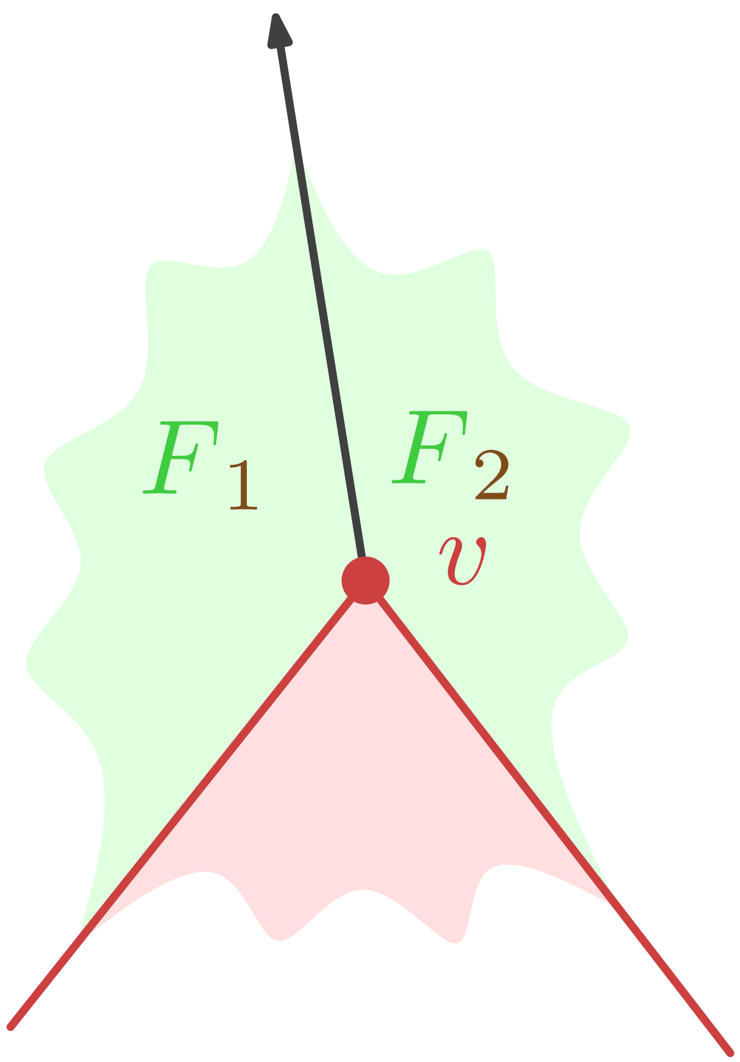

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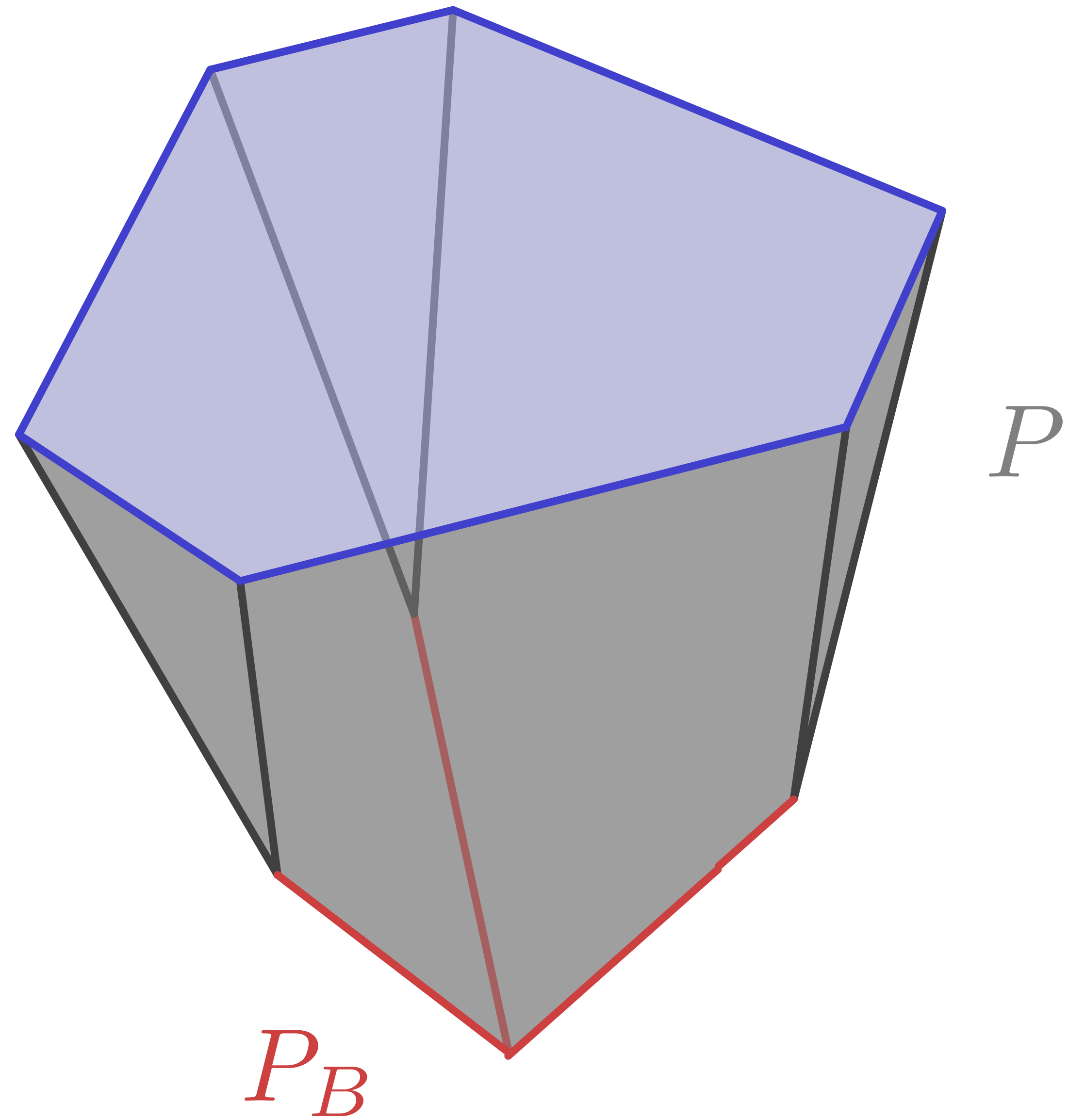

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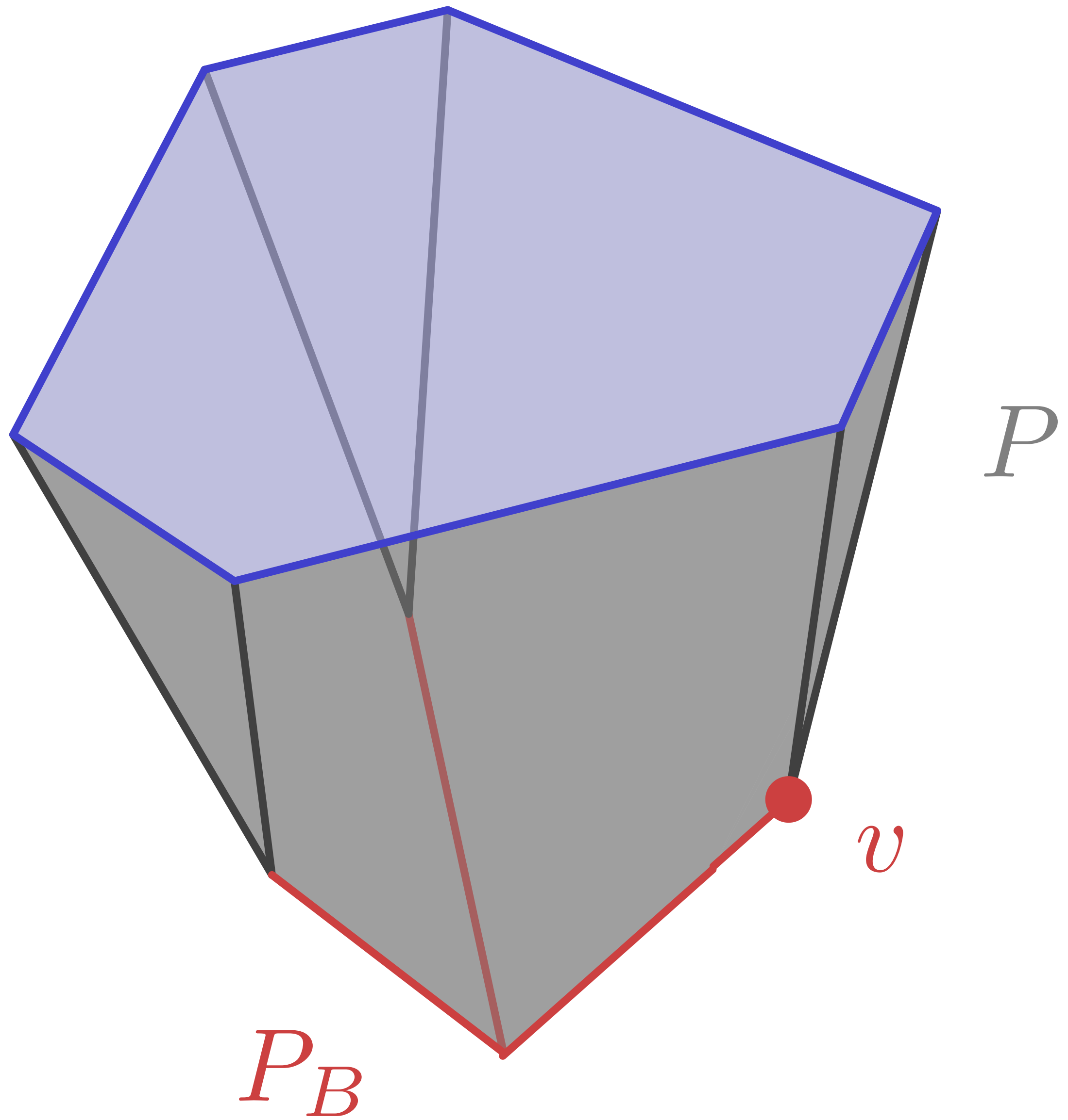

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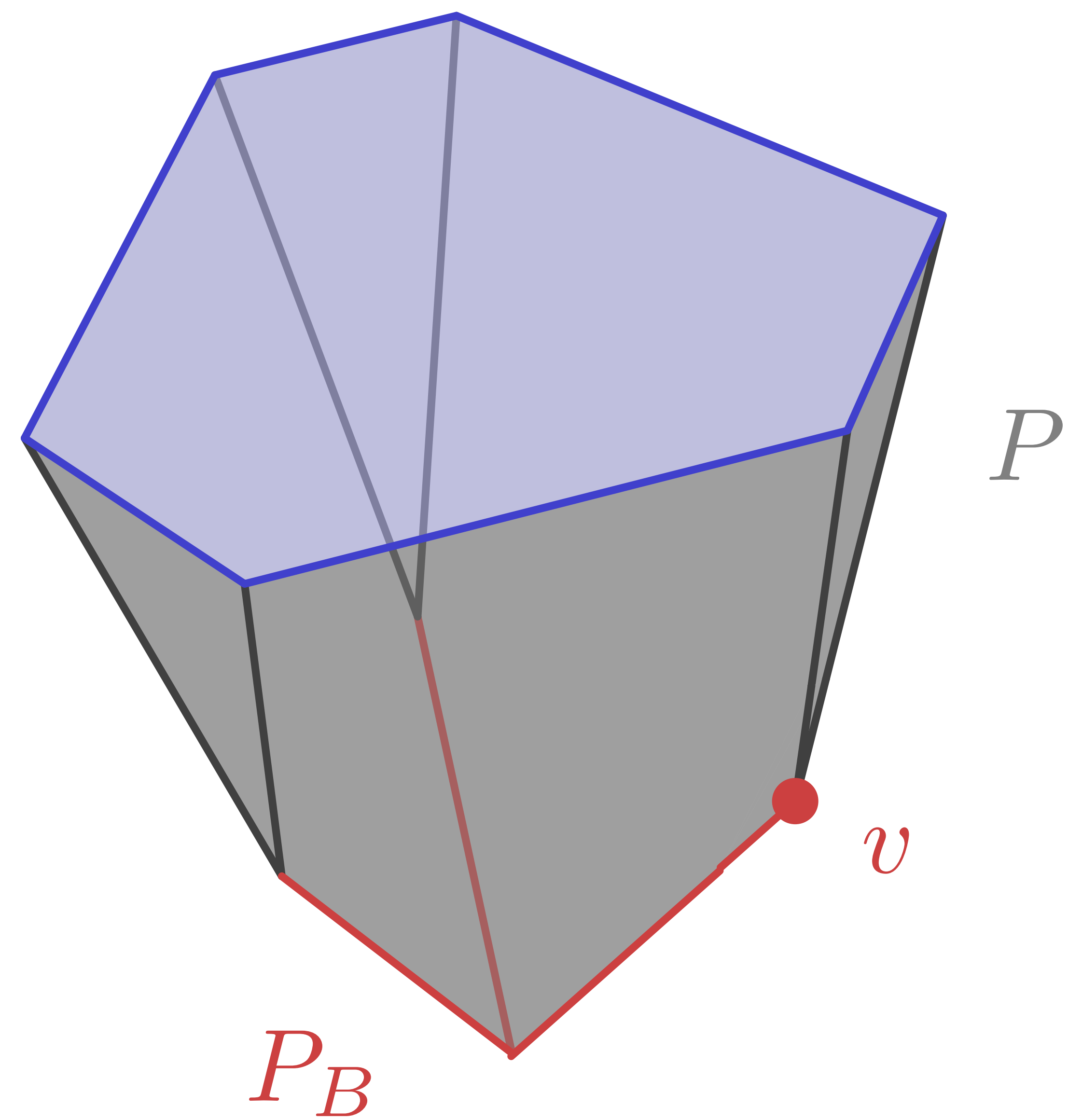

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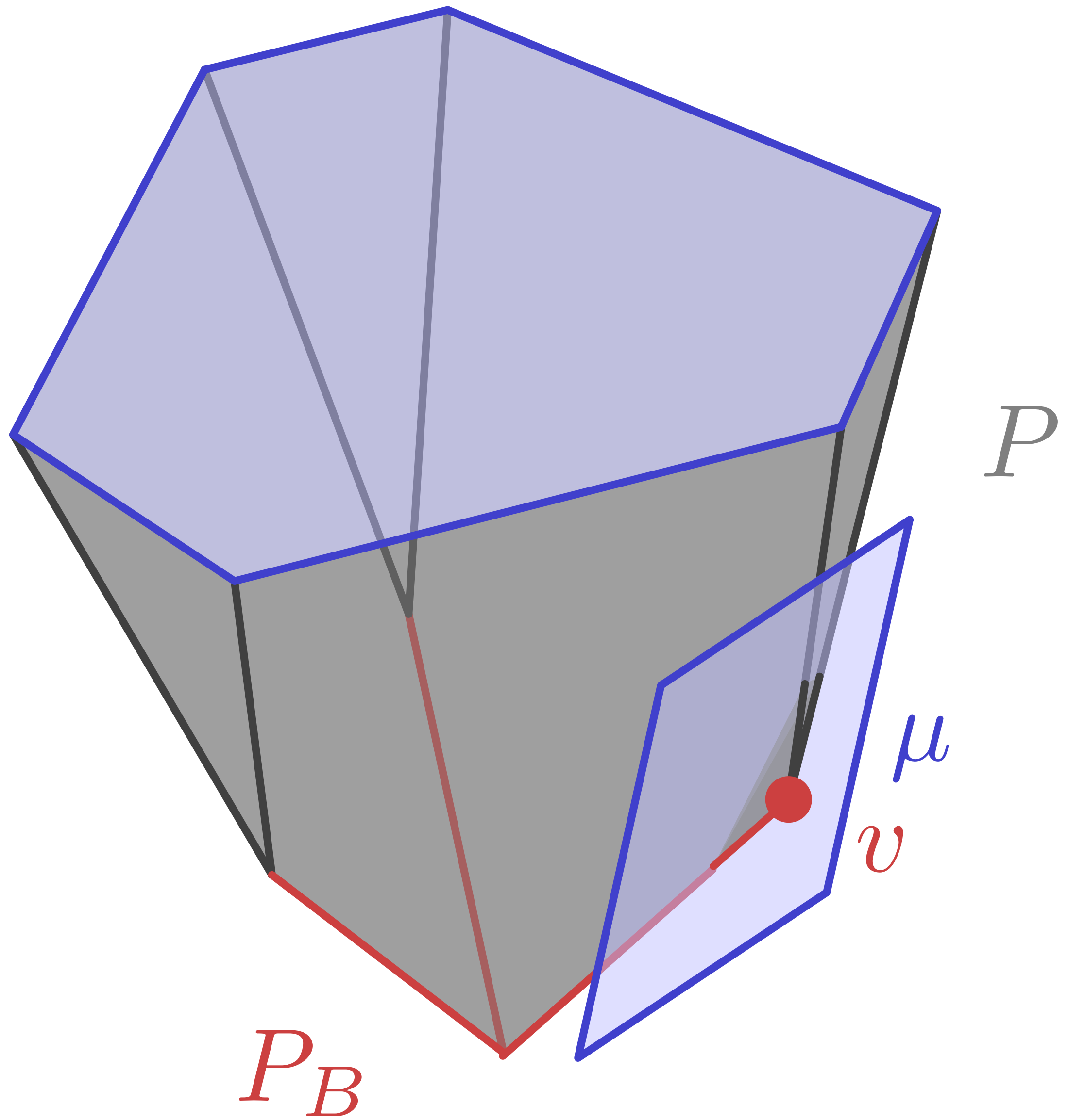

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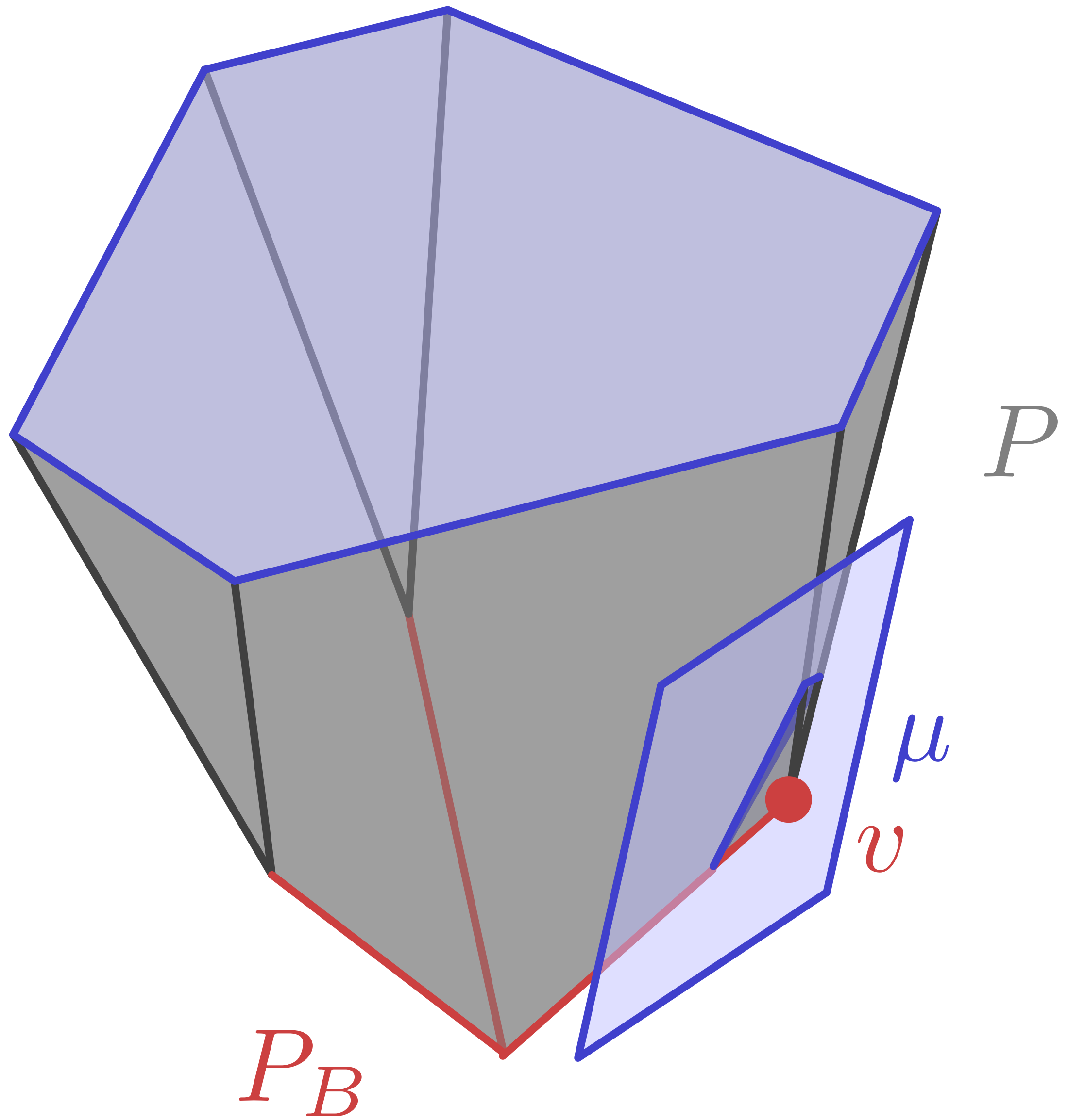

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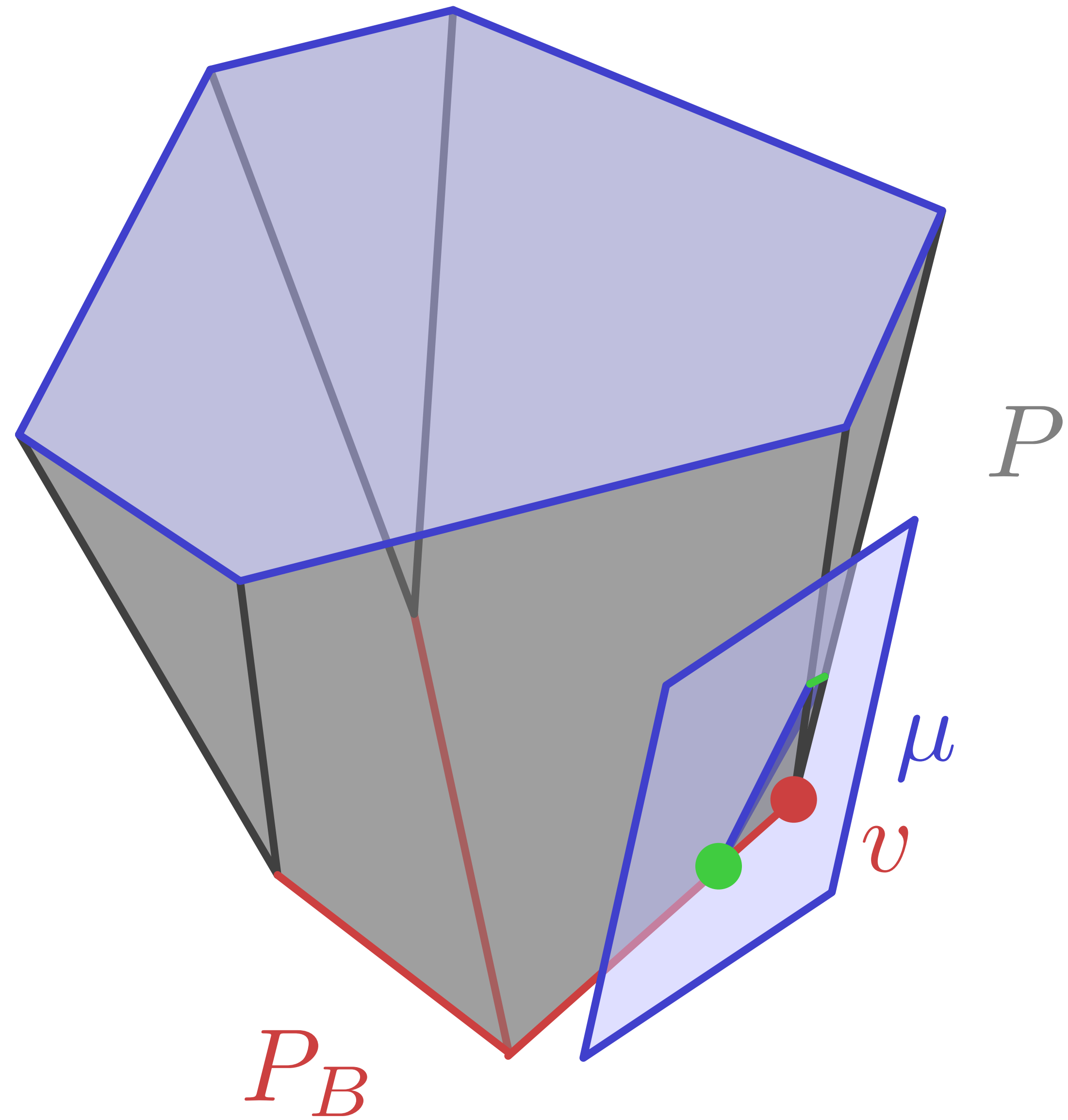

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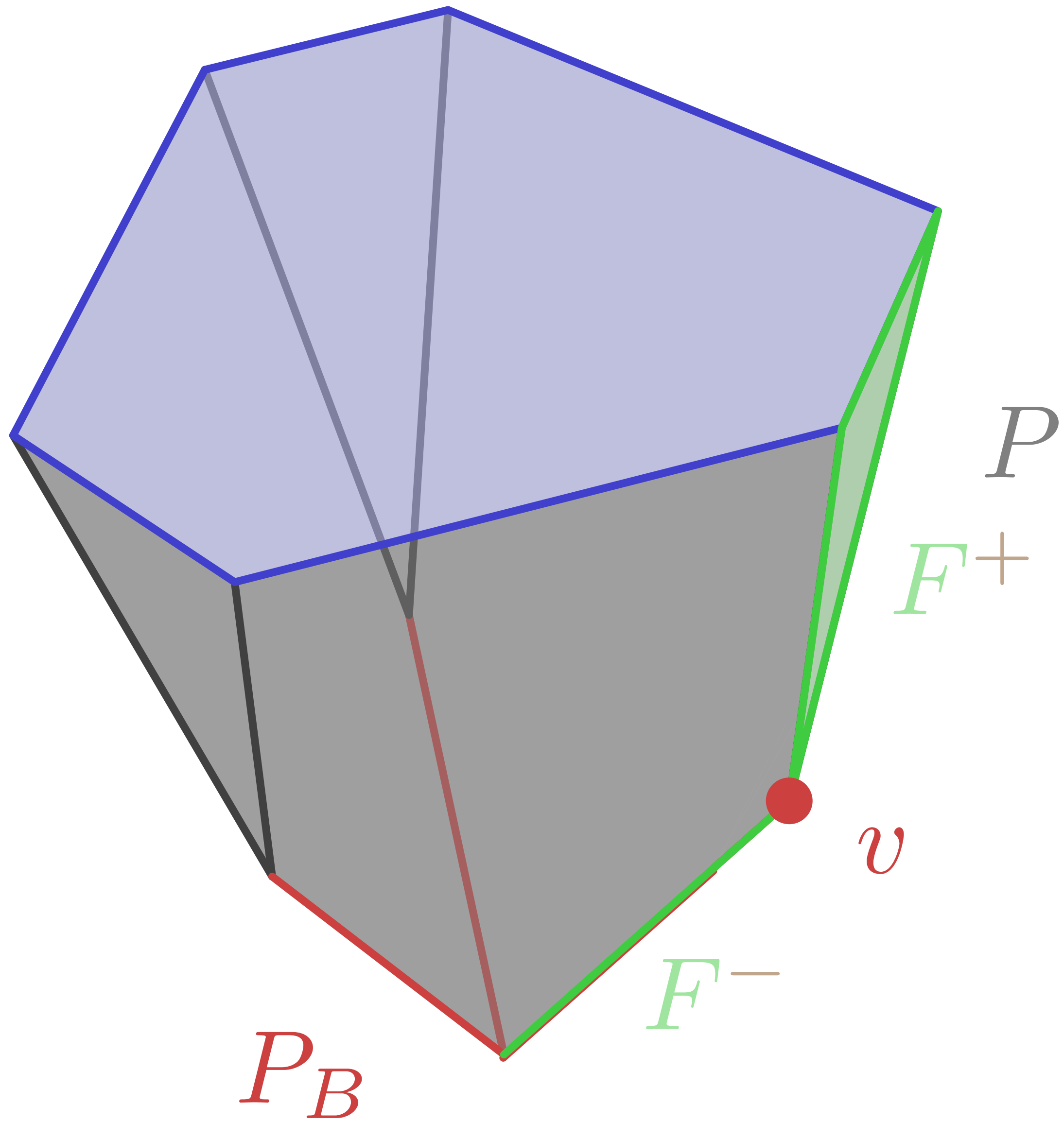

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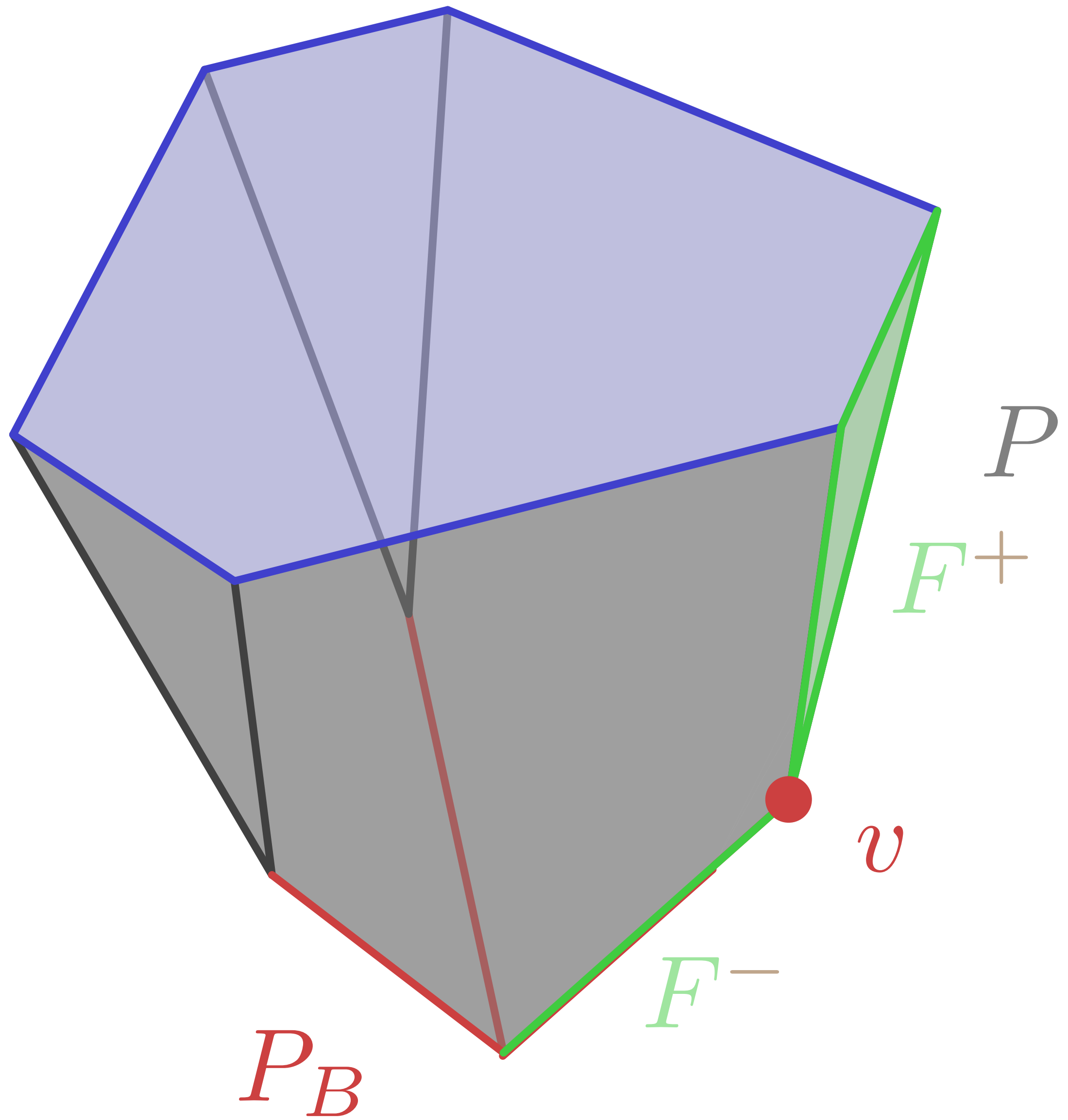
$\dim(F^+) + \dim(F^-) = D + 1$



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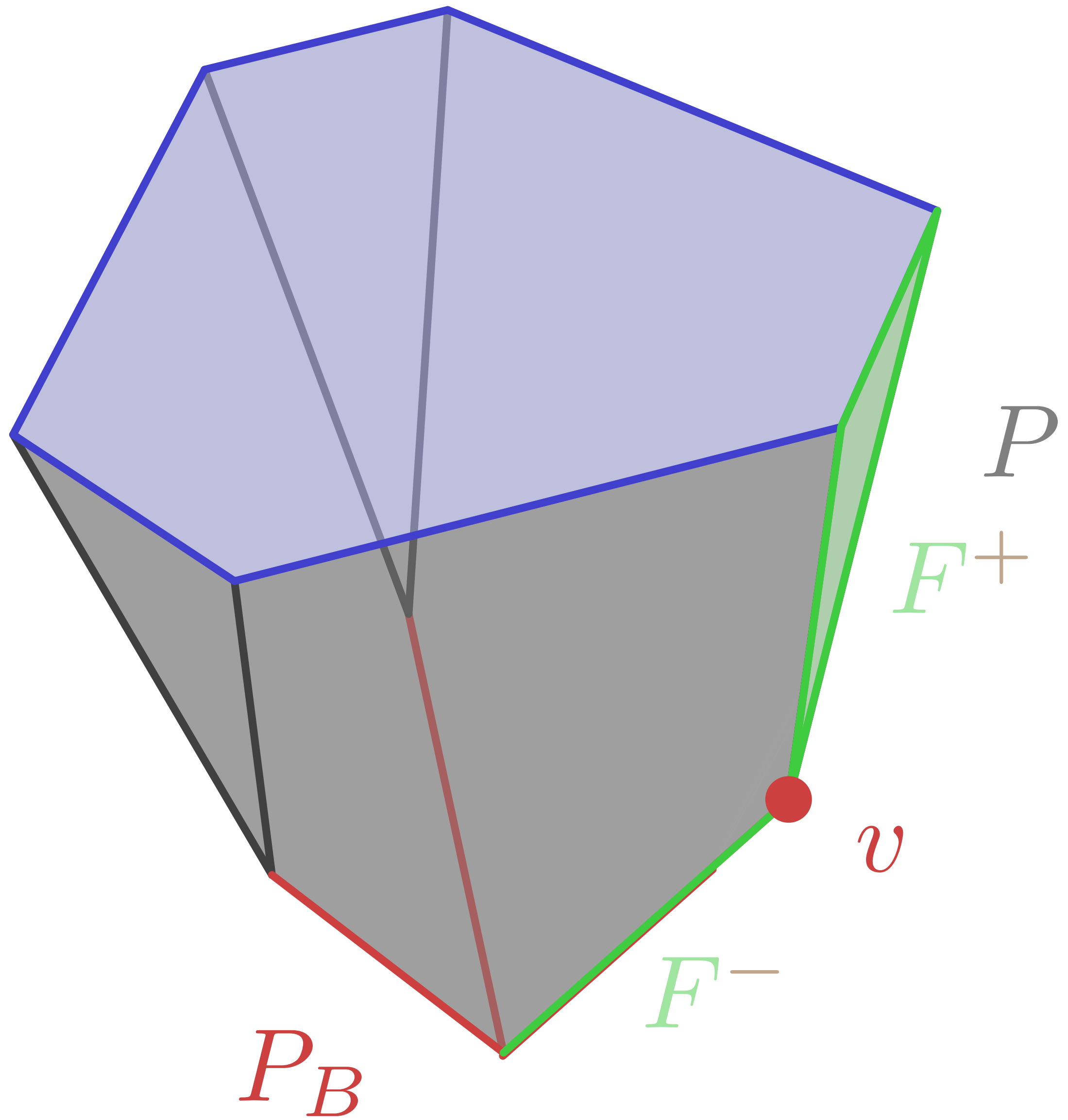

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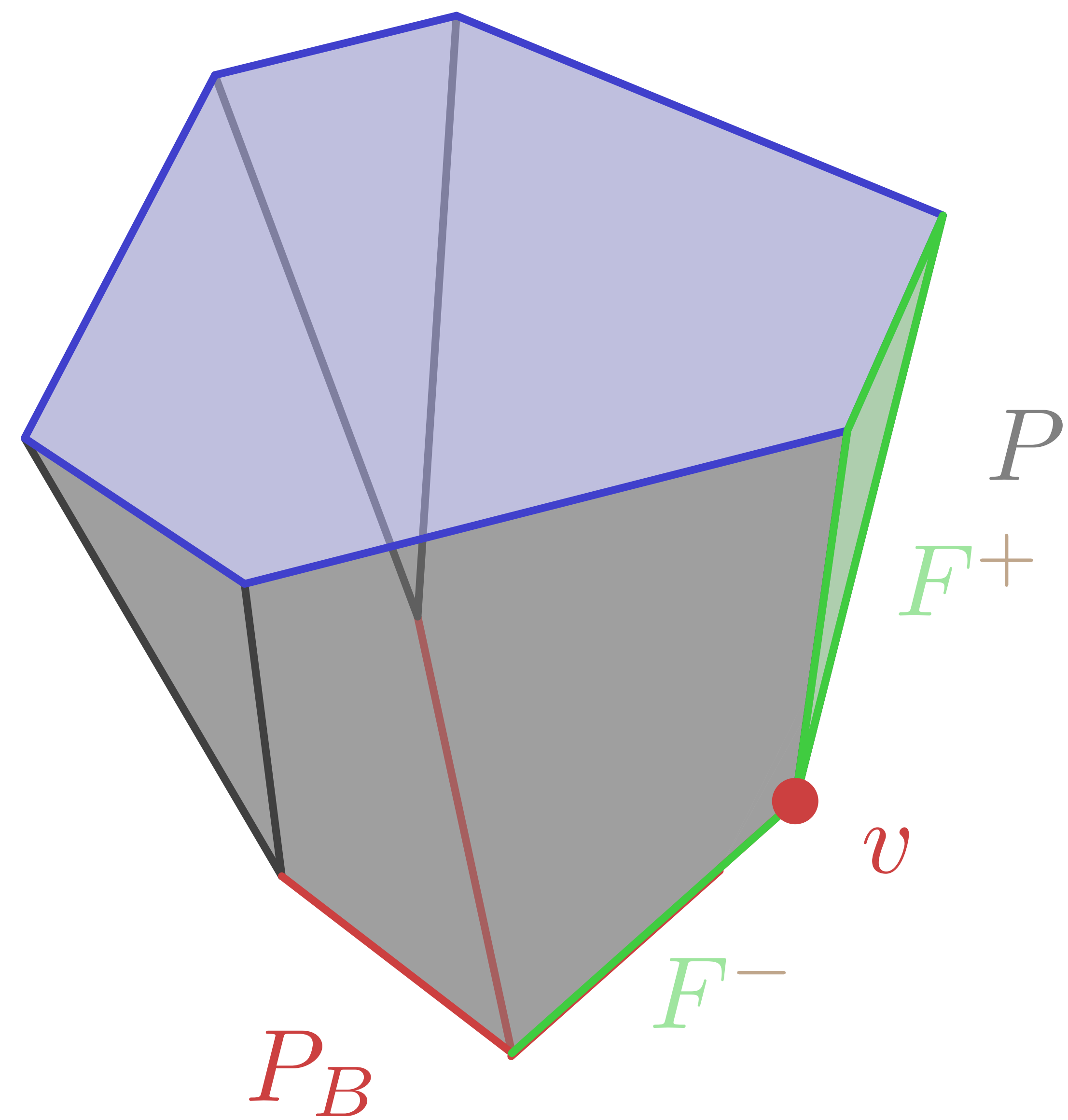

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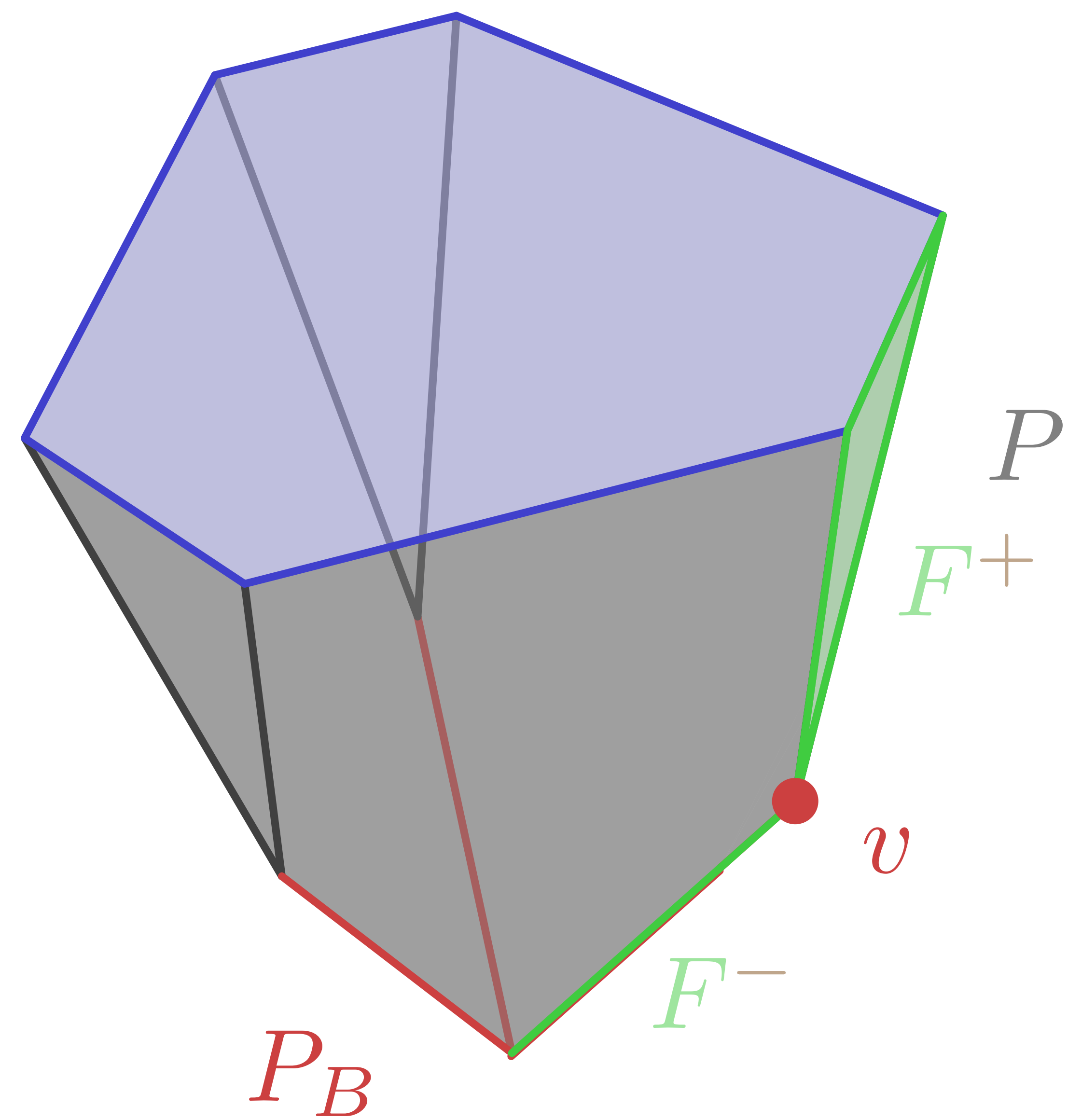
So, F^+ is intersection of d facets.



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(Global minimum was counted $\binom{D}{d}$ times.)

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